MECE 3350U
Control Systems

Lecture 8

Transient Response

Videos in this lecture

Lecture: https://youtu.be/J8jp_3KaXLw

Exercise 35: https://youtu.be/FgjQOuxgwdO

Exercise 36: https://youtu.be/bhnWk-hTjgI

Exercise 37: https://youtu.be/aOJt3LBX1Uw

Exercise 38: https://youtu.be/VdJisStqrx8

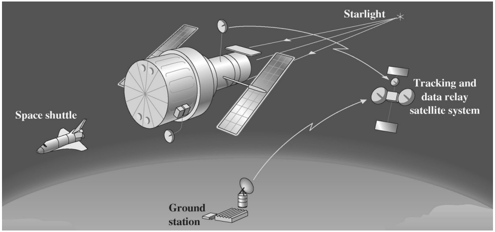
Exercise 39: https://youtu.be/1XTYlE6O4iE

Applications

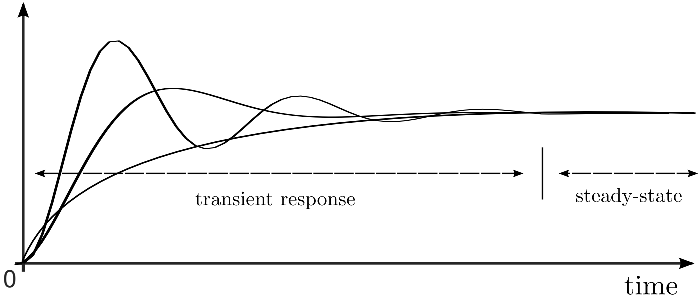
The levitation control system of the train must ensure that the train does not touches the guide. How can we design a controller that reacts as fast as possible with no overshoot?

Applications

The pointing control system of a space telescope is desired to achieve an accuracy of 0.01 minute of arc. How can we limit the steady state error while avoiding transient oscillations?

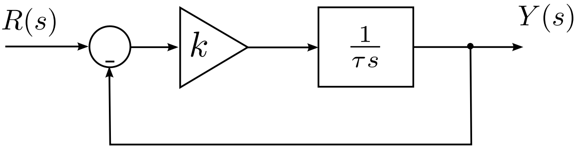


Transient response



First order systems

Consider the first order closed-loop system shown with a proportional gain k



The transfer function $Y(s)/R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{1}{\left(\frac{\tau}{k}\right)s + 1}$$

How does k influence the transient and steady state response?

To analyse the performance of the system, we need to specify a **standard test input signal**.

Standard test signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

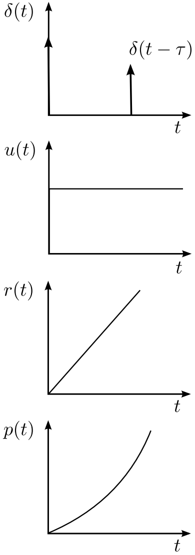
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Temporal response

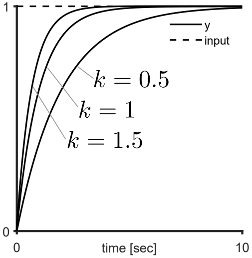
Step response $r(t) = 1$

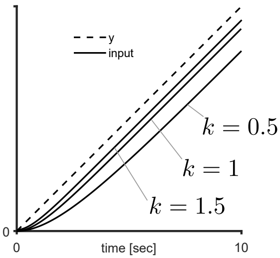
$$Y(s) = \frac{1}{s}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{k}{\tau}t}$$

Ramp response $r(t) = t$

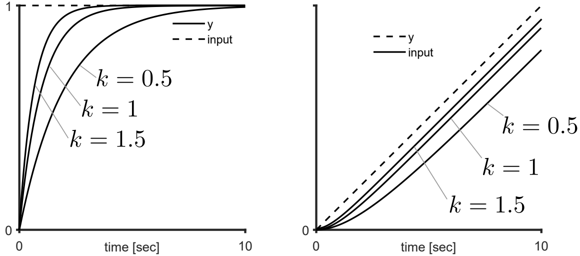
$$Y(s) = \frac{1}{s^2}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = t - \frac{\tau}{k}(1 - e^{-\frac{k}{\tau}t})$$

Effects of k for $k > 0$ for $\tau = 1$.





Temporal response - first order system



In a first-order system:

→ k reduces the **time constant** of the system

→ The higher k, the faster the response

→ What is the maximum control-loop gain k ?

Time constant - first order systems

Impulse:

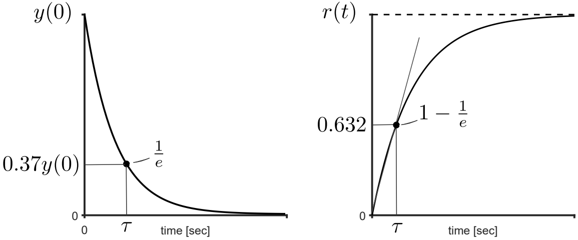
$$H(s) = \frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = y(0)e^{-\frac{t}{\tau}}$$

$\rightarrow$ When $t = \tau$, the response 37% (1/e) of $y(0)$

Step response

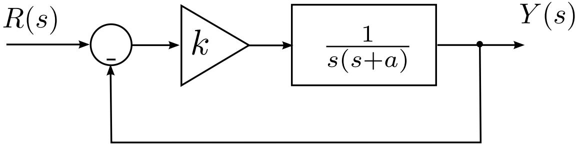
$$H(s) = \frac{1}{s}\frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{t}{\tau}}$$

$\rightarrow$ When $t = \tau$, the response 67% (1-1/e) of its steady state value



Second-order systems

Consider now the following second order control system:



The transfer function is

$$\frac{Y(s)}{R(s)} = \frac{k}{s^2 + sa + k}$$

We can rewrite the above equation in the standard formulation:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

Transient response

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

How does k influence the response of the system?

$\rightarrow$ The natural frequency ω_n depends on k

$\rightarrow$ The damping ratio ζ depends on k

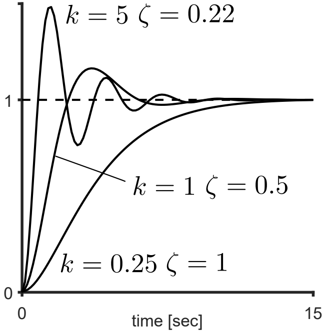
The response for an unit step input when $0 < \zeta < 1$ is

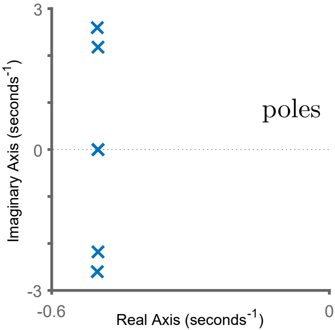
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) t + \cos^{-1}\zeta\right]$$

Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}\zeta\right) \quad (1)$$



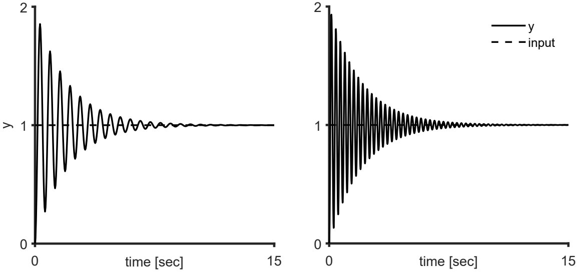


Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

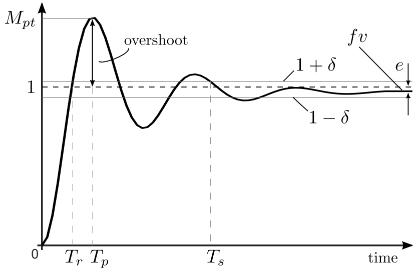
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right)$$

Natural frequency for $k = 1000$ and $k = 5000$.



How can we evaluate the performance of the controller?

Measures of performance



→ Rise time T_r, peak time T_p, and peak value M_{pt}

→ Settling time T_s: $y(t)$ within 2% of its final value

→ Percent overshoot $P.O.$

→ T_r and T_r characterize the **swiftness** of the response

→ $P.O.$ and T_s characterize the **closeness** of the response to the input

Overshoot

For an unit step input, the percent overshoot is

$$P.O. = \frac{M_{pt} - fv}{fv} \times 100 \quad (2)$$

→ M_{pt} is the peak value

→ fv is the magnitude of the input

Differentiating Eq (1) and setting it to zero yields the peak time as

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad (3)$$

Replacing (3) into (1) gives the peak response:

$$M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad (4)$$

Thus, the percentage overshoot ($fv = r(t) = 1$) is

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad (5)$$

Settling time

For an unit step input and $0 < \zeta < 1$, recall that

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right)$$

When $t = T_s$, the response is within 2% of its final value, thus:

$$e^{-\zeta\omega_n T_s} < 0.02$$

or

$$\zeta\omega_n T_s \approx 4 \qquad (6)$$

therefore

$$T_s = \frac{4}{\zeta\omega_n} = 4\tau \qquad (7)$$

where $\tau = 1/\zeta\omega_n$ is the time constant.

The settling time is equal to 4 times the time constant

Settling time

The characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

has poles:

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$
$$s_1 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

or $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. Since

$$T_s \approx \frac{4}{\zeta\omega_n} \tag{8}$$

Therefore the settling time is inversely proportional to the real part of the poles.

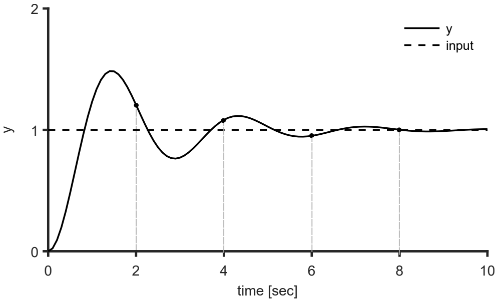
Settling time

Consider the function

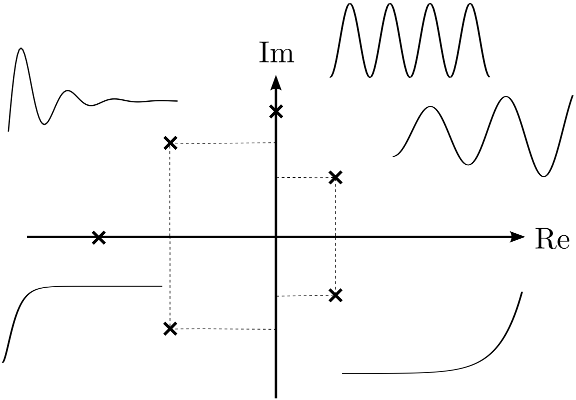
$$H = \frac{5}{s^2 + s + 5}$$

thus: $\omega_n = \sqrt{5}$ and $\zeta = 1/(2\sqrt{5})$. The time constant is

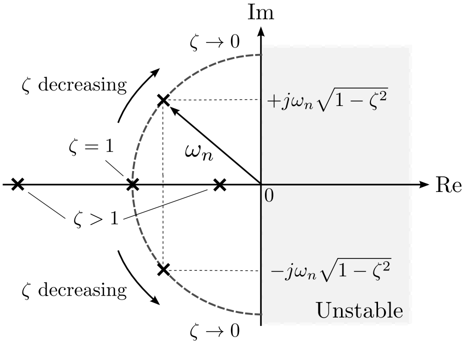
$$\tau = \frac{1}{\zeta\omega_n} = 2 \text{ sec}$$



Poles and transient response



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



Exercise 35

A feedback system with a negative unity feedback has the loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

Determine:

→ **(a)** The closed-loop transfer function

→ **(b)** The time response for an input $r(t) = A$

→ **(c)** The percent overshoot of the response

→ **(d)** The steady state error

Exercise 35 - continued

(a) The closed-loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

Exercise 35 - continued

(b) The time response

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

Exercise 35 - continued

(c) The percentage overshoot

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

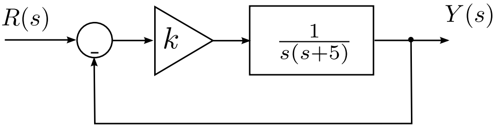
Exercise 35 - continued

(d) The steady state error

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

Exercise 36

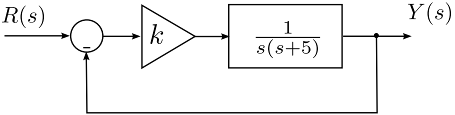
Consider the following block diagram:



→ **(a)** Calculate the steady-state error for a ramp input

→ **(b)** Select k that will result in zero overshoot to a step input

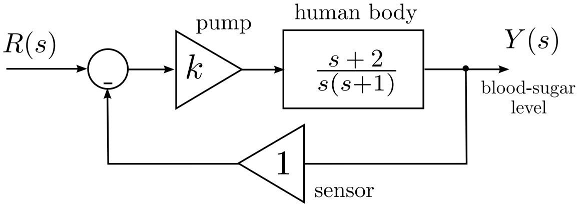
Exercise 36 - continued



Exercise 36 - continued

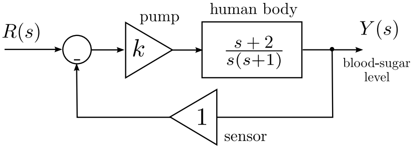
Exercise 37

An insulin pump injection system for diabetic persons has a feedback control as shown.

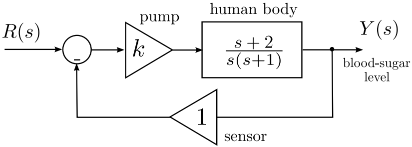


Calculate a suitable gain k so that the percent overshoot of the step response due to the drug injection is 7%. $R(s)$ is the desired blood sugar level and $Y(s)$ is the actual level. Plot the expected overshoot for different k using Matlab.

Exercise 37 - continued

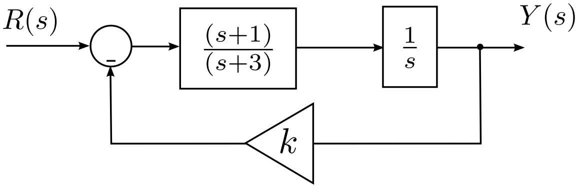


Exercise 37 - continued



Exercise 38

Consider the following closed loop system



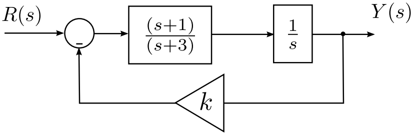
Determine:

→ **(a)** Determine the closed loop transfer function

→ **(b)** Select k so that the steady state error to a unit step input is bounded.

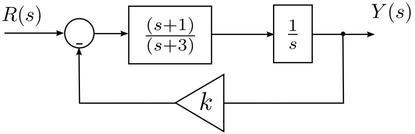
Exercise 38 - continued

(a) Determine the closed loop transfer function



Exercise 38 - continued

(b) Determine the steady state error to a unit ramp input

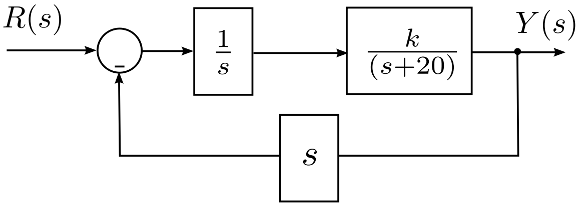


Exercise 38 - continued

(c) Determine k so that $e = 0$ when $r(t) = 1$

Exercise 39

A closed-loop system designed to orient a photovoltaic array towards the direction of maximum solar incidence has the following structure:

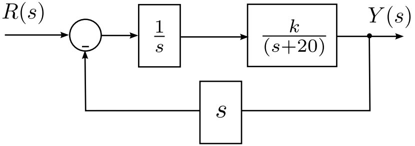


If $k = 20$, determine:

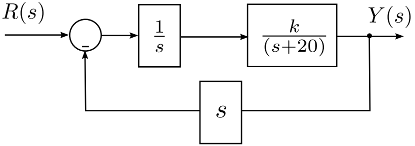
$\rightarrow$ **(a)** The time constant of the closed loop system

$\rightarrow$ **(b)** The settling time to within 2% of the final value of the system to an unit step **disturbance**.

Exercise 39 - continued



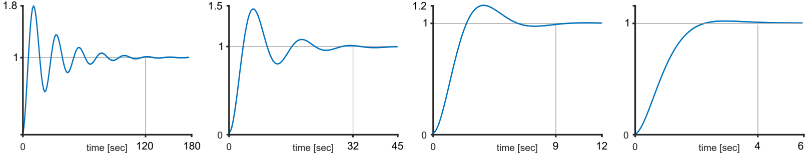
Exercise 39 - continued



Skills check 22 - From 2018 midterm examination

Which of the following time domain signals best describes the response of the function $H(s)$ given below to a unit step-type input? (*4 marks/100*).

$$H(s) = \frac{1}{4s^2 + s + 1}$$

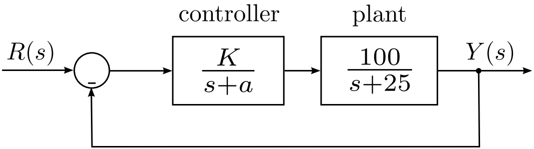


(a) (b) (c) (d)

Answer in the last slides

Skills check 23 - From 2018 final examination

A controller is to be designed for a plant as shown.

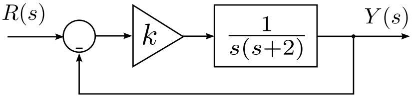


(a) Find the transfer function $Y(s)/R(s)$.

(b) Specify the controller gain K and the value of a so that the overall closed-loop response to a unit step input has an overshoot of no more than 25% and a settling time of no more than 0.1 seconds.

Skills check 24 - From 2018 midterm examination

A robotic system has been designed to operate on a beating heart by ensuring continuous contact between a surgical tool and the heart. The block diagram of such a system is shown below where $R(s)$ is the desired position of the surgical tool i.e., the position of the epicardium (heat wall), and $Y(s)$ is the actual tool position. Any controller overshoot would result in the tool puncturing the heart.



(a) Determine the steady-state error for an unit step input of $R(s)$ as a function of k.

(b) If $k = 10$, what is the expected overshoot ?

(c) Determine the maximum value of k that results in no overshoot.

(d) What is the settling time as a function of k ?

Answers to skills check

SC 22 - (b)

SC 23 - In (a) the transfer function is

$$\frac{Y(s)}{R(s)} = \frac{100k}{s^2 + s(a + 25) + 25a + 100k}$$

For question (b), we get $k \approx 85$ and $a \approx 55$.

SC 24 - (a) $e_s s = 0 \; \forall k$,
(b) P.O. $= 35\%$,
(c) $k < 1$,
(d) $T_s = 4$ sec $\forall k$

Next class...

- Dominant poles