MECE 3350U
Control Systems

Lecture 5
Effect of Pole Locations

Videos in this lecture

Lecture 5: https://youtu.be/vWO-LgqIk2Q

Exercise 21: https://youtu.be/wSlh_tKHOhA

Exercise 22: https://youtu.be/XPbNiesErkU

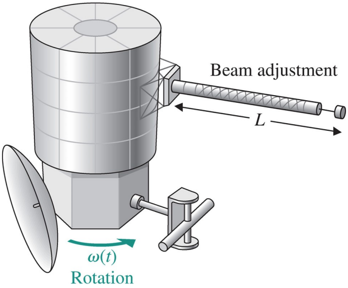
Outline of Lecture 5

By the end of today's lecture you should be able to

- Understand the concept of transient response
- Observe the influence of the pole locations in the temporal response
- Find the temporal response of a system for a given input

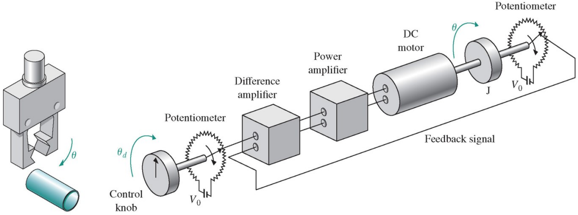
Applications

The rotational velocity of of the satellite is adjusted by changing the length of the beam. How can we determine the shape of the transient response?



Applications

A robot gripper is to be controlled by a DC motor. How can we determine the transient response of the gripper's position?



From the last lecture

A transfer function can be written as

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \quad (1)$$

with $n \geq m$. The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\lim_{s \to z_i} N(s) = 0 \quad (2)$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

$$\lim_{s \to p_i} H(s) = \infty \quad (3)$$

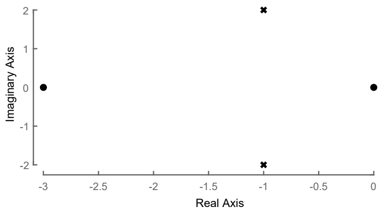
Poles and zeros

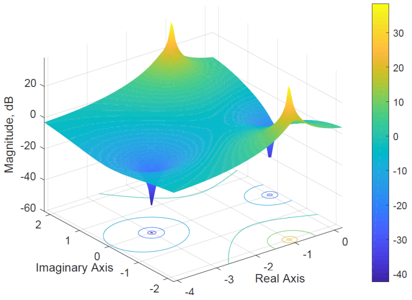
Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2+2s+5}$$

→ Poles: $-1+2j,\ -1-2j$

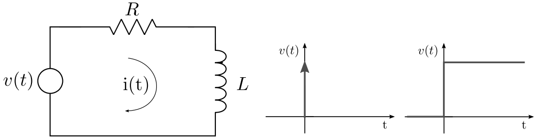
→ Zeros: $0,\ -3$





First order systems

Consider the RL circuit shown.



$$V(s) = (R + Ls)I(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1/R}{s(L/R) + 1}$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \tag{4}$$

$\rightarrow$ The denominator must be in the form of $\tau s + \mathbf{1}$

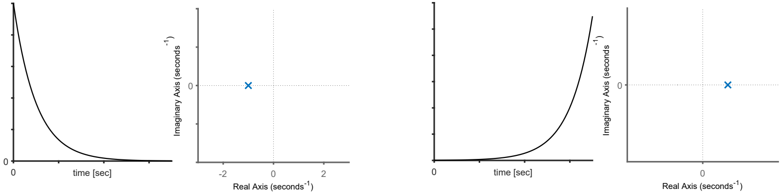
First order transfer functions

Impulse response: $v(t) = \delta(t) \Rightarrow V(s) = 1$

$$I(s) = \frac{1}{R}\frac{1}{\tau s + 1} = \frac{1}{\tau R}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

The pole is $s = -1/\tau$. The time response is:

$$i(t) = \frac{1}{\tau R} e^{-\frac{t}{\tau}}$$



If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

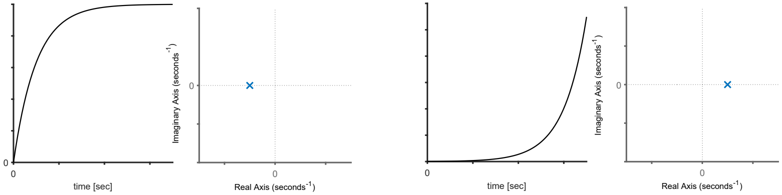
First order transfer functions

Step response: $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$I(s) = \frac{1}{\tau R}\left(\frac{1}{s}\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{k_1}{s} + \frac{k_2}{s+\frac{1}{\tau}}\right)$$

Solving for the partial fraction coefficients: $k_1 = \tau$, $k_2 = -\tau$, thus:

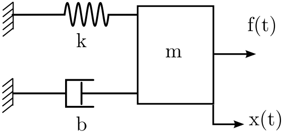
$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

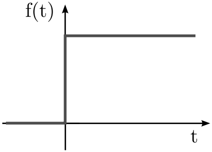


If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

Second order transfer functions





Transfer function: Standard form

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{m}\left(\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\right)$$

In standard form we have

$$H(s) = \frac{1}{k}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

where:

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

Second order response

Let us now analyse the response to a step input of a second order system

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{6}$$

The poles of the transfer function are:

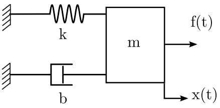
$$s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{7}$$

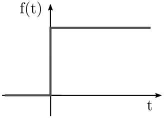
Thus:

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\zeta + j\sqrt{1 - \zeta^2}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\zeta - j\sqrt{1 - \zeta^2}\right)$$

Roots can be real or complex $\Rightarrow$ Inverse transformation ?

Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{8}$$

Case 1: $\zeta \geq 1$, (lots of damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

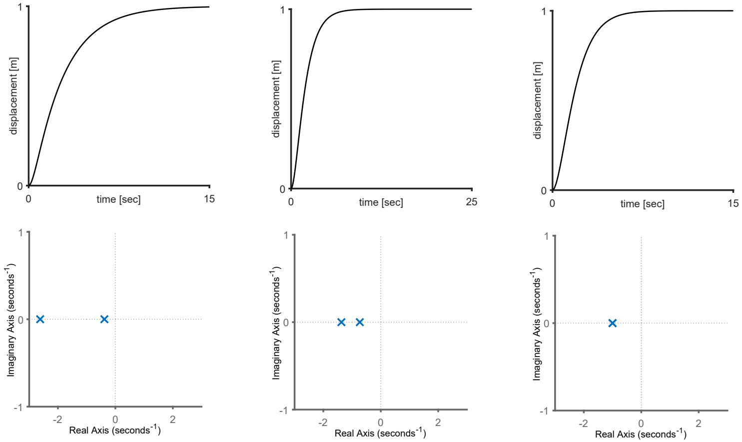
Roots are negative real numbers. Partial fraction expansion yields:

$$X(s) = \frac{1}{m}\left(\frac{k_1}{s} + \frac{k_2}{(s + a_1)} + \frac{k_3}{(s + a_2)}\right) \tag{9}$$

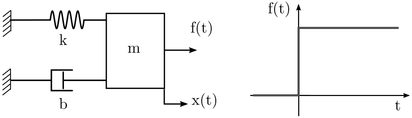
Overdamped system

Example: $m = 1$ kg, $k = 1$ N/m

$b = 3$ Ns/m, $\zeta = 1.5$; $b = 2.1$ Ns/m, $\zeta = 1.05$; $b = 2$ Ns/m, $\zeta = 1$.



Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{10}$$

Case 2: $0 < \zeta < 1$, (some damping)

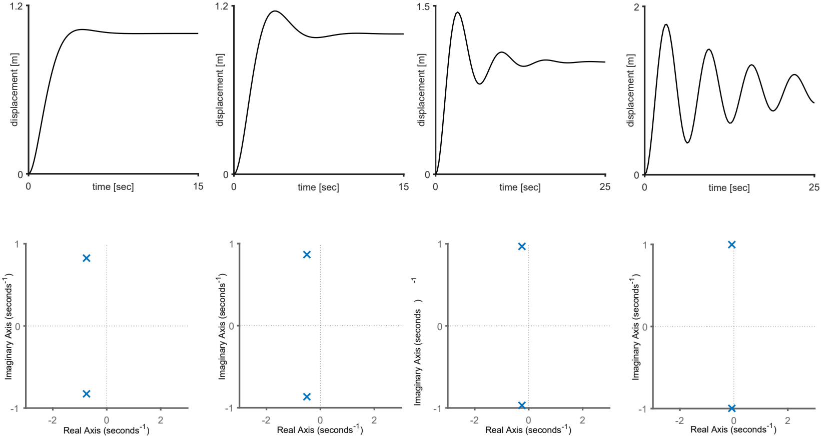
$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

Roots are complex conjugate numbers with a negative real part. Thus:

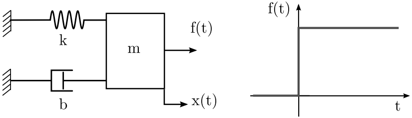
$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \tag{11}$$

Underdamped system

$b = 1.5,\ \zeta = 0.75$ $b = 1,\ \zeta = 0.5;$ $b = 0.5,\ \zeta = 0.25,$ $b = 0.1,\ \zeta = 0.05$



Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{12}$$

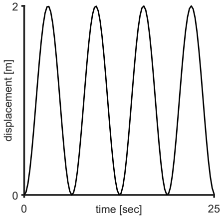
Case 3: $\zeta = 0$, (no damping)

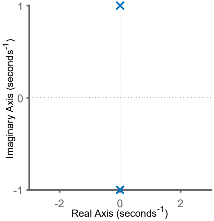
$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

Roots are purely complex conjugate numbers. Thus:

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) \rightarrow x(t) = \frac{1}{k}[1 - \cos(\omega_n t)] \tag{13}$$

Undamped system





The frequency of oscillation for of an undamped system is called the natural frequency.

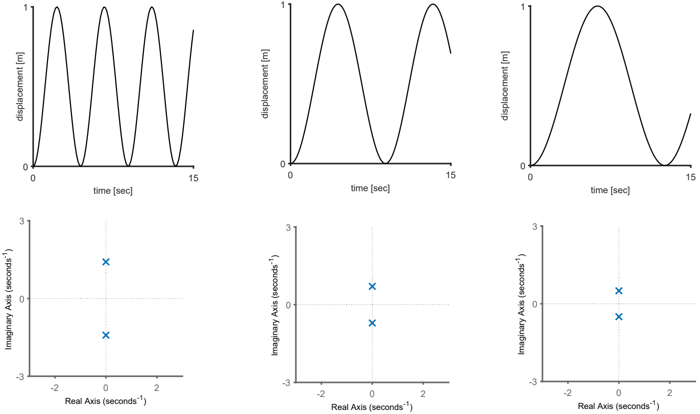
In our example:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{14}$$

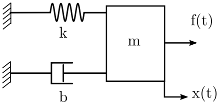
Natural frequency

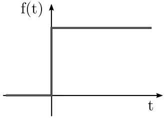
The frequency the system oscillates when $\zeta = 0$. Example: $b = \zeta = 0$, $k = 1$.

$m = 1$, $\omega_n = 1$ rad/s; $m = 2$, $\omega_n = 0.71$ rad/s; $m = 4$, $\omega_n = 0.5$ rad/s;



Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \quad (15)$$

Case 4: $\zeta < 0$, (hypothetical negative damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right), \quad s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

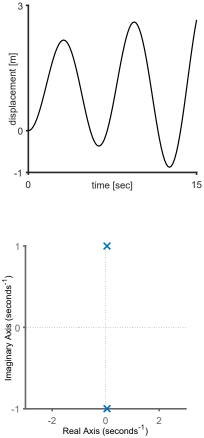
Roots (real or imaginary) have positive real parts. Possible solutions are:

$$x(t) = k(1 + k_2 e^{s_1 t} + k_3 e^{s_2 t}) \quad (16)$$

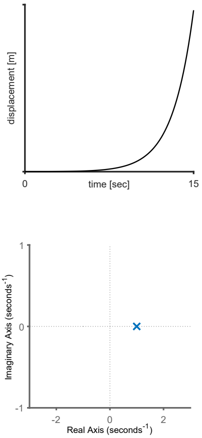
$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{|\zeta|\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \quad (17)$$

Unstable system

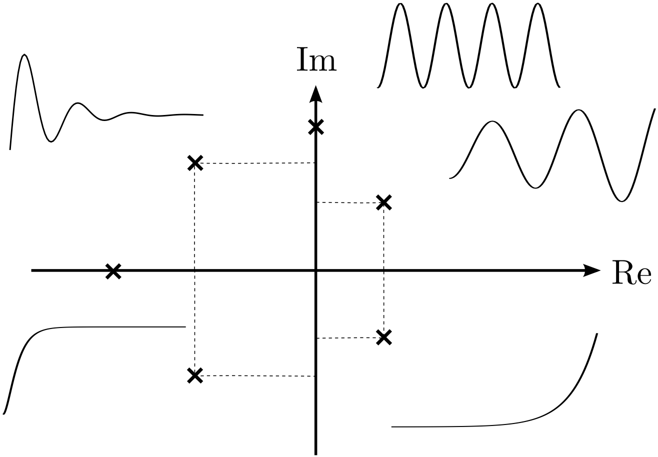
$b = -0.1$ Ns/m;



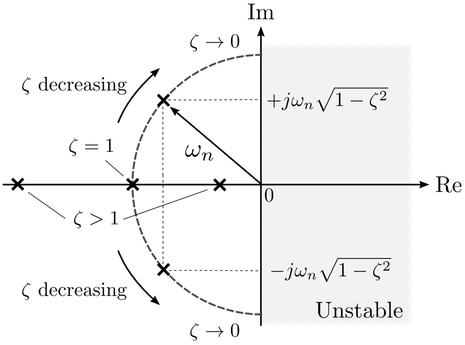
$b < -1$ Ns/m;



Summary



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



Location of poles in the s-plane

For a second order system, the poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

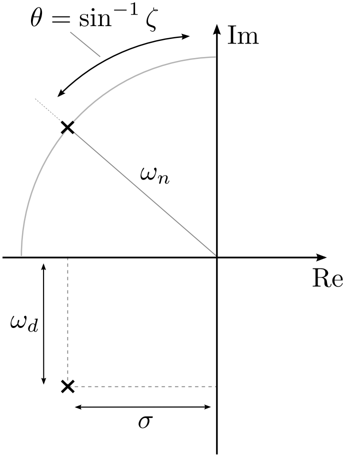
where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

→ The magnitude of s is

$$|s| = \sqrt{(\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2} = \omega_n$$

→ The angle to the imaginary axis is

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} \rightarrow \theta = \sin^{-1}\zeta$$



Exercise 21

Discuss the correlation between the poles of

$$H(s) = \frac{2s+1}{s^2+2s+5} \tag{18}$$

and the impulse response of the system and find the exact impulse response.

Procedure:

$\rightarrow$ Calculate the damping ratio and the natural frequency

$\rightarrow$ Calculate inverse transform

Exercise 21 - continued

$$H(s) = \frac{2s + 1}{s^2 + 2s + 5}$$

Exercise 21 - continued

Exercise 21 - continued

Exercise 22

Consider the system of the form

$$H(s) = \frac{9}{s^2 + bs + 9} \tag{19}$$

Calculate and sketch the response to an unit step input for the following cases

$\rightarrow b = 9$

$\rightarrow b = 0$

$\rightarrow b = 2$

$\rightarrow b = 6$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 9s + 9}\frac{1}{s} \quad (20)$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2+9}\frac{1}{s} \tag{21}$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 2s + 9}\frac{1}{s} \tag{22}$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 6s + 9}\frac{1}{s} \quad (23)$$

Skills check 11 - From 2018 midterm

Consider the following statements regarding a system described by a second order transfer function subjected to a unit step input (*4 marks /100*):

→ Statement 1: If the damping ratio is higher than 1, there is no overshoot.

→ Statement 2: If the real part of a zero is positive, the system is unstable.

→ Statement 3: If the real part of both poles is zero, the system oscillates indefinitely.

Statements 1, 2, and 3 are, respectively[1]

(a) All false
(b) True, false, and false
(c) False, false, and true
(d) True, false, and true
(e) False, true, and false
(f) True, true, and false
(g) False, true, and true
(h) All true

[1] Answers to skills check on slide 40

Skills check 12 - From 2018 midterm

Indicate which transfer function leads to each of the time responses shown for a unit step-type excitation (*4 marks/100*)

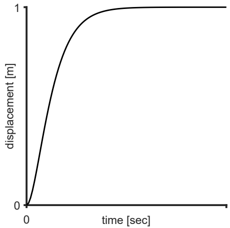
(a) $\frac{2}{(s+1)(s+1)}$

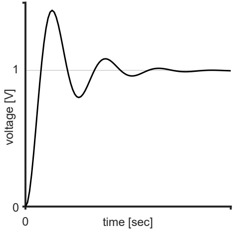
(b) $\frac{1}{s^2+1}$

(c) $\frac{1}{2s^2+s+1}$

(d) $\frac{s}{s^2+0.5s+1}$

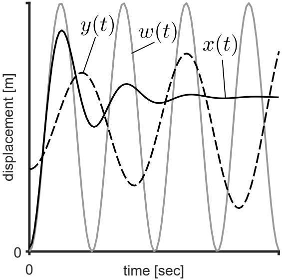
(e) $\frac{1}{2s+1}$

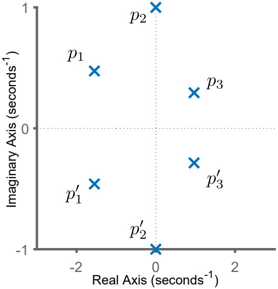




Skills check 13

The poles of the three functions are shown in the s-plane. Circle the correct statement(s)





(a) p_1, p_1' are the poles of $Y(s)$

(b) p_1, p_1' are the poles of $W(s)$

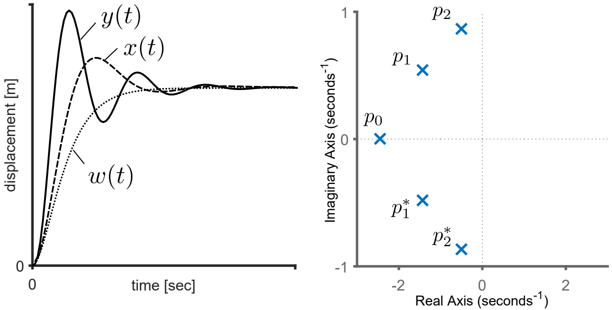
(c) p_2, p_2' is the pole of $W(s)$

(d) p_3, p_3' are the poles of $W(s)$

(e) p_3, p_3' are the poles of $Y(s)$

Skills check 14

The poles of the three functions are shown in the s-plane. Circle the correct statement(s)



(a) p_1, p_1^* are the poles of $Y(s)$

(b) p_1, p_1^* are the poles of $X(s)$

(c) p_0 is the pole of $X(s)$

(d) p_2, p_2^* are the poles of $X(s)$

(e) All the above statements are false

Skills check 15

Which transfer function leads to the time response shown for a unit step-type excitation?

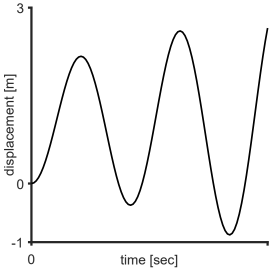
(a) $\frac{1}{s^2-0.05s+1}$

(b) $\frac{1}{s^2+0.05s+1}$

(c) $\frac{1}{s^2-5s+1}$

(d) $\frac{1}{s-2}$

(e) $\frac{1}{s+2s+5}$



Skills check 16

The time response of a second order system converges to a steady-state value provided that

(a) All poles have positive real parts

(b) All poles have negative real parts

(c) All zeros have positive real parts

(d) All poles have positive imaginary parts

(e) (a) and (d) are required

Answers to skills check

SC 11 - (d)

SC 12 - Left graph corresponds to (e) (why not (a)?)
Right graph corresponds to (c)

SC 13 - Only statements (c) and (e) are correct

SC 14 - Only statement (b) is correct

SC 15 - (a) is correct. Why not (c)?

SC 16 - (b)

Next class...

- Block diagrams