MECE 3350U
Control Systems

Lecture 4

Transfer Functions

Videos in this lecture

Lecture 4: `https://youtu.be/KP6DATDCIdc`

Exercise 16: `https://youtu.be/2BBO3lcdm5U`

Exercise 17: `https://youtu.be/DoL2rarunKg`

Exercise 18: `https://youtu.be/Oo8Woo0oIhs`

Exercise 19: `https://youtu.be/HZMWeGR0RFo`

Exercise 20: `https://youtu.be/bz7P2R0XYMw`

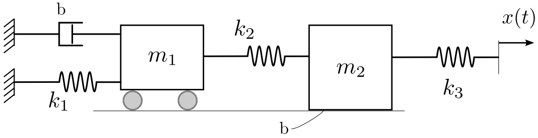
Outline of Lecture 4

By the end of today's lecture you should be able to

- Understand the concept of transfer functions
- Find the transfer function of a given system
- Find the temporal response of a system for a given input

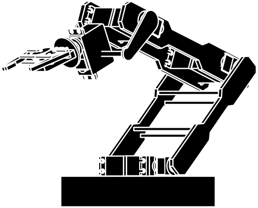
Applications

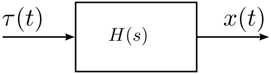
If the spring is stretched to a point $x(t) = 5$ mm, held, then released at time $t = 0$, how does the position of m_1 evolve in time?



Applications

What torque must be applied to each of the robot joints so that end-effector moves along a given trajectory with a given speed?

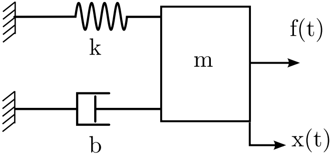




Input/output relation

Transfer function: A relation between the input and output of a linear system

Example: Input: force $f(t)$, output: displacement $x(t)$



$$m\frac{d^2x}{dt} + b\frac{dx}{dt} + kx = f(t)$$

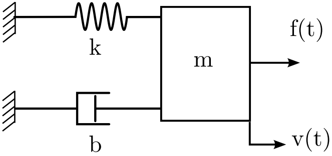
Taking the Laplace transform:

The transfer function is

$$H(s) = \frac{X(s)}{F(s)} = \tag{1}$$

Input/output relation

Example: Input: force $f(t)$, output: **velocity** $v(t)$



The dynamic model is

$$ \tag{2} $$

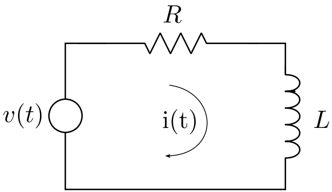
Laplace transform of (2) is

The transfer function is

$$H(s) = \frac{V(s)}{F(s)} = \tag{3}$$

Input/output relation

Input: Voltage $v(t)$, output: Current $i(t)$



$$v(t) = Ri(t) + L\frac{di(t)}{dt} \tag{4}$$

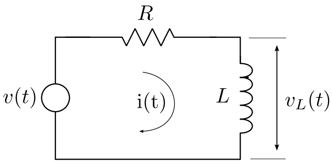
Taking the Laplace transform of (4):

The transfer function is

$$H(s) = \frac{I(s)}{V(s)} =$$

Input/output relation

Input: Voltage $v(t)$, **output**: $v_L(t)$



$$v(t) = Ri(t) + L\frac{di(t)}{dt}, \qquad v_L = L\frac{di(t)}{dt} \tag{5}$$

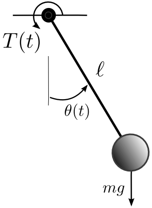
Taking the Laplace transform of (5):

The transfer function is

$$\frac{I(s)}{V(s)} = \frac{1}{Ls+R}, \qquad \frac{V_L(s)}{V(s)} =$$

Input/output relation

The equation of motion of the simple pendulum is



$$m\frac{d^2\theta(t)}{dt^2} + m\frac{g}{\ell}\sin\theta = T(t) \tag{6}$$

For small angles, $\sin\theta \approx \theta$, in the frequency domain:

$$\frac{\theta(s)}{T(s)} = \tag{7}$$

Transfer function poles and zeros

Transfer function: A rational function in the complex variable $s = \sigma + j\omega$:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \tag{8}$$

The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\lim_{s \to z_i} N(s) = 0 \tag{9}$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

$$\lim_{s \to p_i} H(s) = \infty \tag{10}$$

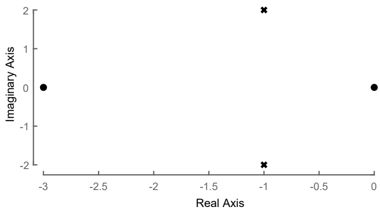
Poles and zeros

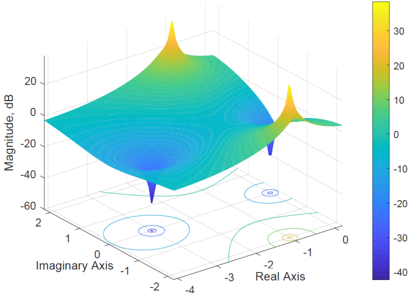
Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2+2s+5}$$

→ Poles: $-1+2j$, $-1-2j$

→ Zeros: 0, -3





First order transfer functions

Order: The number of the highest derivative in the denominator (power of s)

Standard form of a first order system:

$$G(s) = k\frac{1}{\tau s + 1}$$

Characteristic equation: The denominator of the transfer function

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} =$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \qquad (11)$$

$\rightarrow$ The denominator must be in the form of $\tau s + \mathbf{1}$

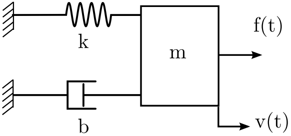
Second order transfer functions

Standard form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

Where: ω_n is the natural frequency, ζ is the damping ratio.

We will come back to these definitions in the next lecture.



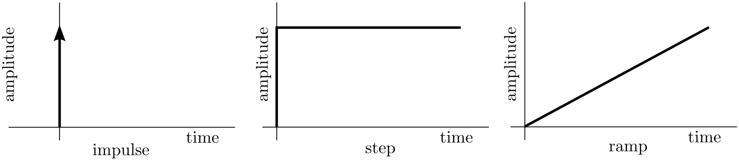
$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} =$$

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

Temporal response

Step 1: Define the input signal in Laplace domain

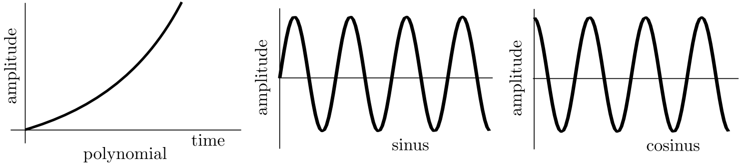


input signal	time domain	frequency domain
impulse	$r(t) = \delta(t)$	$R(s) = 1$
step	$r(t) = a$	$R(s) = a\frac{1}{s}$
ramp	$r(t) = at$	$R(t) = a\frac{1}{s^2}$

a is a constant

Temporal response

Step 1: Define the input signal in Laplace domain



input signal	time domain	frequency domain
polynomial	$r(t) = at^n$	$R(s) = a\frac{n!}{s^{n+1}}$
sine	$r(t) = \sin(at)$	$R(s) = \frac{a}{s^2+a^2}$
cosine	$r(t) = \cos(at)$	$R(t) = \frac{s}{s^2+a^2}$

a is a constant

Temporal response

Step 2: Replace the input signal in the transfer function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{13}$$

For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k} \tag{14}$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right) \tag{15}$$

For a sinusoidal input $f(t) = 5\sin(t)$ N:

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{5}{s^2 + 1}\right) \tag{16}$$

Temporal response

Step 3: Calculate the inverse transform of the resulting function

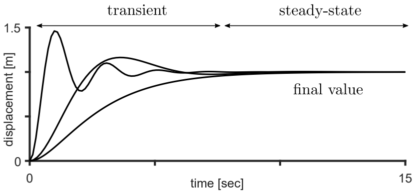
For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\right\} \tag{17}$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right)\right\} \tag{18}$$

and so on.



Steady-state value

Final value theorem: Gives the steady-state value without computing the inverse transform.

If the function converges, i.e., the poles of $sX(s)$ have negative real parts, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{19}$$

For a step type input, the mass spring damper system settles at

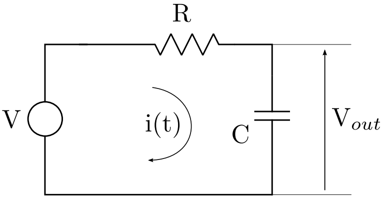
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left\{\frac{1}{ms^2+bs+k}\left(\frac{1}{s}\right)\right\} = \tag{20}$$

For a impulse input, the RL system converges at

$$\lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s) = \lim_{s\to 0} s\left\{\frac{1}{Ls+R}\right\} = \tag{21}$$

Exercise 16

Find the transfer function $H(s)$ between the input voltage V and the output voltage V_{out}.



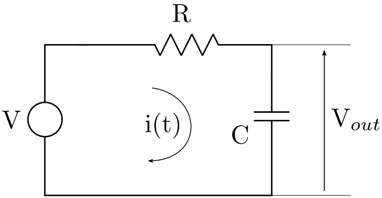
Procedure:

→ Find the differential equation for the current

→ Find the equation for the output voltage

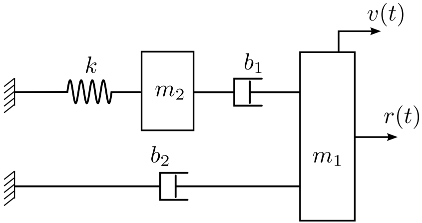
→ Calculate the transfer function

Exercise 16 - continued



Exercise 17

Find the transfer function $H(s) = \frac{V(s)}{R(s)}$ between the force $r(t)$ and the velocity of mass m_1.



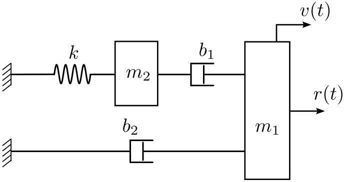
Procedure:

$\rightarrow$ Find the differential equation the velocity of each mass

$\rightarrow$ Calculate the Laplace transform

$\rightarrow$ Calculate the transfer function

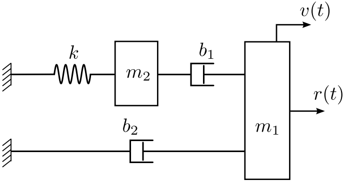
Exercise 17 - continued



Exercise 17 - continued

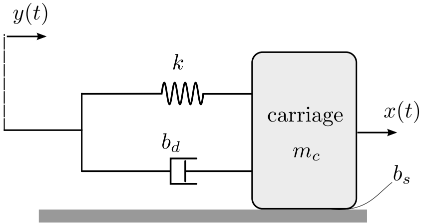
$$[m_1 s + (b_1 + b_2)]V(s) - b_1 V_2(s) = R(s)$$

$$-b_1 V(s) + \left(m_2 s + b_1 + \frac{k}{s}\right) V_2(s) = 0$$



Exercise 18

A high precision positioning slide is shown in the figure. The drive shaft friction is $b_d = 0.65$, the drive shaft spring constant is $k = 1.8$, $m_c = 1$, and the slide friction is $b_s = 0.9$.



Determine:

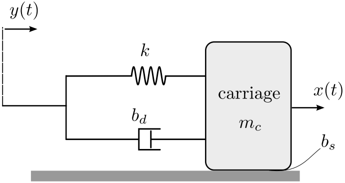
→ Find the transfer function $H(s) = X(s)/Y(s)$.

→ Calculate the natural frequency, damping ratio, the poles, and zeros of $H(s)$

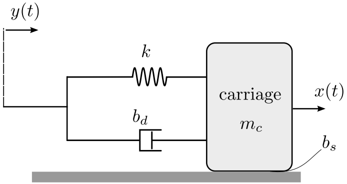
→ Find the steady-state value for a step input

→ Plot the step response of using Matlab

Exercise 18 - continued

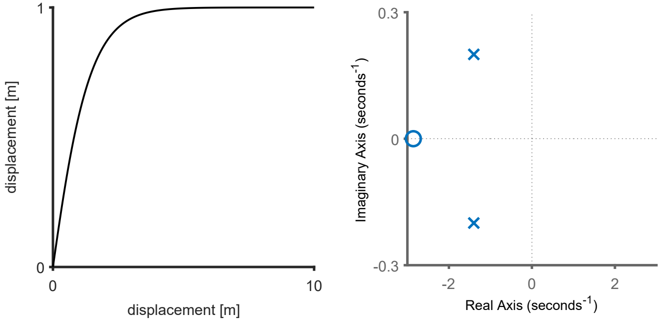


Exercise 18 - continued



Exercise 18 - continued - Using Matlab

H = tf([0.7 2],[1 2.8 2]) → Transfer function
damp(H) → Natural frequency and damping
step(H,10) → Step response
pzplot(H) → Location of zeros and poles



Exercise 19

Calculate the natural frequency and damping ratio of the following transfer function

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

Determine:

→ Write the transfer function in standard form

→ Find the steady-state value for a step input

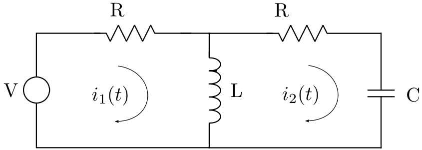
→ Calculate the natural frequency and damping ratio

Exercise 19 - continued

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

Exercise 20

Find the transfer function $G(s) = \frac{I_2(s)}{V(s)}$ of the circuit shown. Then, calculate the step response of the circuit using Matlab. Take $R = 10\ \Omega$, $C = 0.001$ F, $L = 0.1$ H, $V = 5$ V.



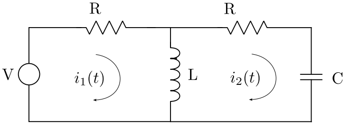
Determine:

→ Find the transfer function $H(s) = I_2(s)/V(s)$.

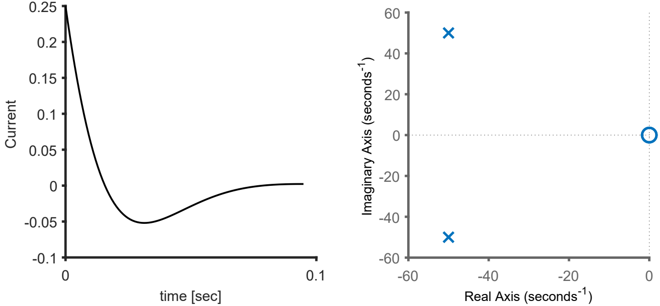
→ Find the steady-state value for a step input

→ Plot the step response of using Matlab

Exercise 20 - continued

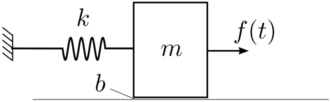


Exercise 20 - continued - Using Matlab



Skills check 5 - From 2018 midterm

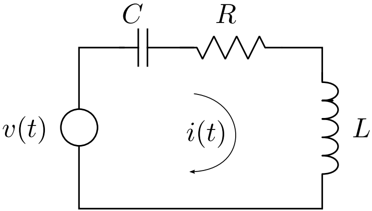
Find the transfer function between the acceleration $A(s)$ and the force $F(s)$ of the mass-spring system shown. The coefficient of viscous friction between the mass and the plane is b (*4 marks*)[1].



[1] $H(s) = s^2/(ms^2 + bs + k)$

Skills check 6 - From 2018 final examination

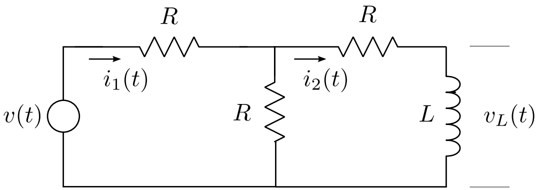
Find the transfer function between the input voltage $v(t)$ and the **electric charge** $q(t)$ in the LRC circuit (*3 marks out of 100*)[2].



[2] $H(s) = 1/[L(s^2 + (R/L)s + 1/(LC)]$

Skills check 7 - From 2018 final examination

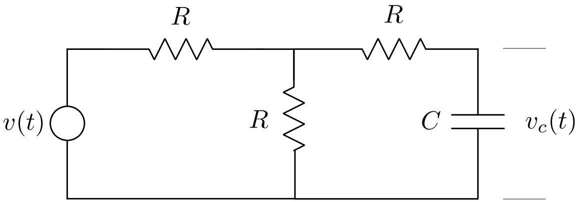
Find the transfer function between the input voltage $V(s)$ and the voltage across the inductor $V_L(s)$[3].



[3] Consult instructor or TA for answer

Skills check 8 - Strongly recommended

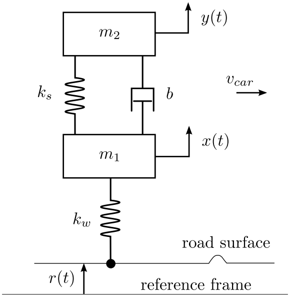
Consider the RC network shown, where $v(t)$ is the input voltage and $v_c(t)$ is the circuit output voltage. R is the same for all resistors. Using the transfer function and its inverse Laplace transform, determine the temporal response of $v_c(t)$ for a **unit step-type input** of $v(t)$.[4]



[4] $v(t) = 0.5(1 - e^{-2t/(3RC)})$

Skills check 9 - From 2018 assignment 1

The figure shows an automobile suspension system composed of two masses m_1 and m_2, a spring of stiffness k_s, and a damper with a coefficient of viscous friction b. The lower spring k_w represents the tire compressibility[5].



(a) Find the transfer function $Y(s)/R(s)$.

(b) Calculate the final value of $y(t)$ for a unit step-type displacement of the road surface, i.e., $r(t) = 1$.

[5] Answer see next slide

Skills check 9 - Answer

(a)

$$\frac{Y(s)}{R(s)} = \frac{k_w b}{m_1 m_2}\left(\frac{s + \frac{k_s}{b}}{s^4 + as^3 + bs^2 + cs + d}\right) \tag{22}$$

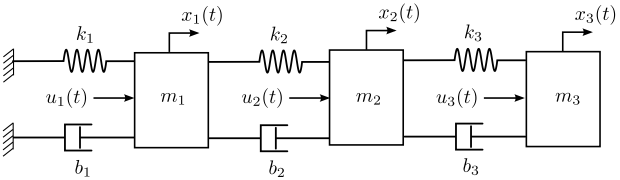
where

$a = \frac{b}{m_1} + \frac{b}{m_2}$
$b = \frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}$
$c = \frac{k_w b}{m_1 m_2}$
$d = \frac{k_w k_s}{m_1 m_2}$

(b) $\lim_{t\to\infty} y(t) = 1$

Skills check 10

Consider the three-mass system shown. The systems has three input forces i.e., $u_1(t)$, $u_2(t)$, $u_3(t)$, and three outputs $x_1(t)$, $x_2(t)$, and $x_3(t)$.



Obtain the equations of motion in the frequency domain. All initial conditions are zero[6].

[6] Final answer in the next slide

Skills check 10 - Answer

For mass 1

$$U_1(s) + (k_2 + sb_2)X_2(s) = [m_1 s^2 + (b_1 + b_2)s + k_1 + k_2]X_1(s)$$

For mass 2

$$U_2(s) + (k_3 + sb_3)X_3(s) + (k_2 + b_2 s)X_1(s) = [m_2 s^2 + (b_2 + b_3)s + k_2 + k_3]X_2(s)$$

For mass 3

$$U_3(s) + (k_3 + sb_3)X_2(s) = (m_3 s^2 + b_3 s + k_3)X_3(s)$$

Next class...

- Effect of pole locations