MECE 3350U
Control Systems

Lecture 3

Laplace Transform

Videos in this lecture

Lecture 3: https://youtu.be/qadBOxaiHzw

Exercise 11: https://youtu.be/p9NJo7mT-cc

Exercise 12: https://youtu.be/zgbq8AALvbU

Exercise 13: https://youtu.be/ka6P3yYA-JQ

Exercise 14: https://youtu.be/5Mke8kokKDI

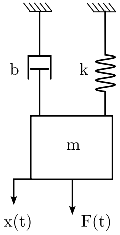
Exercise 15: https://youtu.be/7EPAV6XvASM

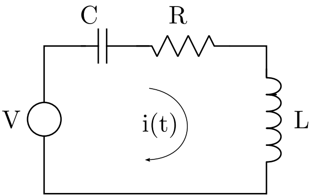
Outline of Lecture 3

In today's lecture we will

- Review the principles of the Laplace transformation
- Apply the Laplace transformation to a system

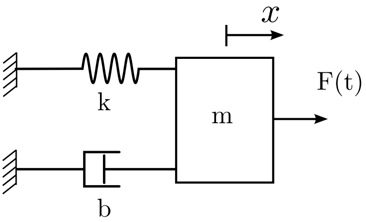
Applications

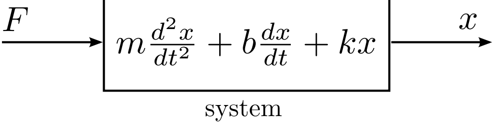




Input/output relation

Transfer function: A relation between the input and output of a given linear system

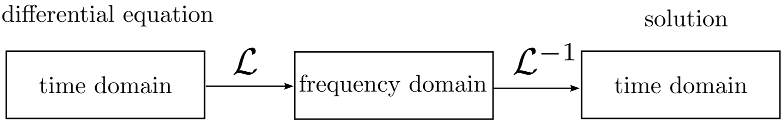




How can we evaluate the temporal response to the system?

Laplace transformation

The time-response solution can be obtained using the Laplace transform.



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \qquad F(s) = X(s)[ms^2 + bs + k] \qquad x(t) = Ke^{-\alpha t}\sin(\beta t + \theta)$$

$\rightarrow$ Obtain the linearised differential equations

$\rightarrow$ Obtain the Laplace transformation of the differential equation

$\rightarrow$ Solve for the variable of interest

Laplace transformation

A mass-spring system is governed by the differential equation

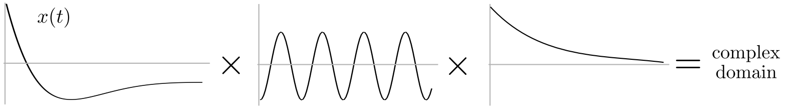
$$m\frac{d^2x(t)}{dt^2} + kx(t) = F(t) \tag{1}$$

Solutions of (1) can be

→ Exponential: $x(t) = e^{-\sigma t}$ with $\sigma \in \mathcal{R}$

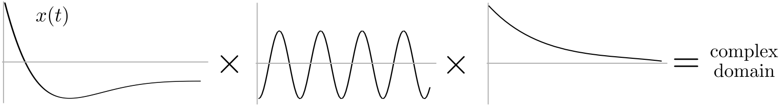
→ Sinusoidal: $x(t) = \sin(\omega t) = e^{-j\omega t}$ with $j \in \mathcal{C}$

→ Exponential and sinusoidal: $x(t) = e^{-j\omega t}e^{-\sigma t}$



Laplace transformation

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} \left[x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t}\right] dt$$



The standard form of the Laplace transform is:

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} \left[x(t) \cdot e^{(-\sigma - j\omega)t}\right] dt \qquad (2)$$

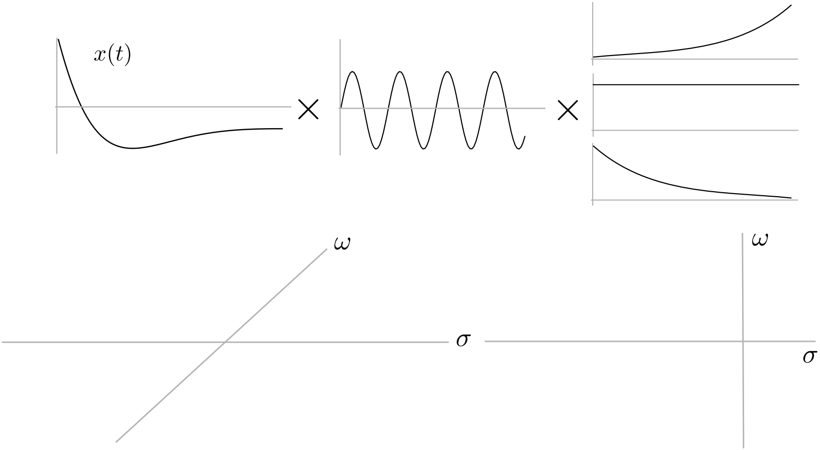
where the complex variable $s = \sigma + j\omega$ has:

→ A real portion σ, which corresponds to the exponential response

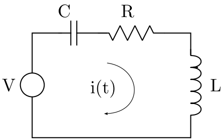
→ An imaginary portion ω, which corresponds to the sinusoidal response

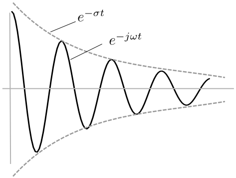
Laplace transformation

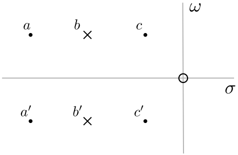
$$X(\sigma,\omega) = \int_{-\infty}^{\infty} \left[x(t) \cdot e^{(-\sigma - j\omega)t} \right] dt \tag{3}$$

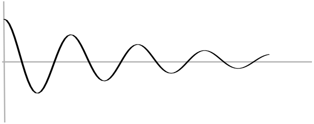


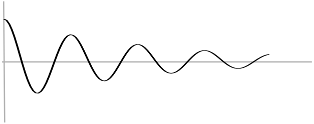
Laplace transformation

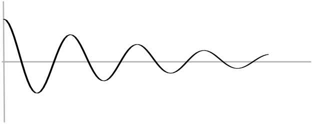


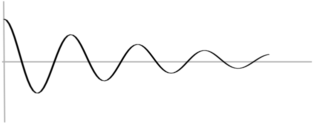




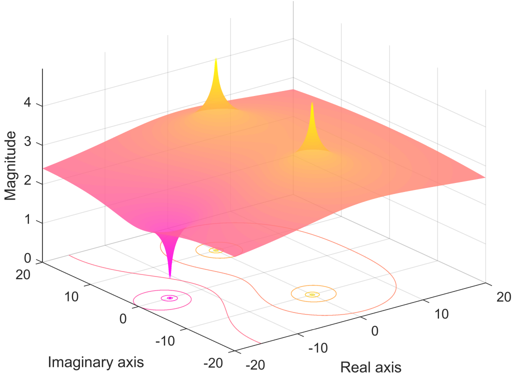








What is the Laplace transform?



Laplace transformation

The Laplace transformation for a function of time $f(t)$ is

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \mathscr{L}\{f(t)\} \tag{4}$$

where $s = \sigma + j\omega$.

The inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j}\int_{\tau - j\infty}^{\tau + j\infty} F(s)e^{st}ds \tag{5}$$

Transfer function: The ratio of the Laplace transform of the output variable of to the input variable.

Laplace transformation

What is the Laplace transform of $f(t) = 1$?

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathscr{L}\{f(t)\}$$

Laplace transformation

What is the Laplace transform of $\cos(5t)$?

Laplace transformation

$f(t)$	$F(s)$
Impulse function $\delta(t)$	1
Step function $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace transformation

$f(t)$	$F(s)$
$\frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2} \dot{f}(0) - ... - f^{k-1}(0)$
$\int f(t) dt$	$\frac{F(s)}{s} + \frac{1}{s} \int f(t) dt \Big\|_{t=0}$

The Laplace variable s can be considered to be the differential operator:

$$s \Rightarrow \frac{d}{dt} \tag{6}$$

And the integral operator:

$$\frac{1}{s} \Rightarrow \int_0^t dt \tag{7}$$

Laplace transform properties

Linearity

$$\mathscr{L}\{\alpha x(t)\} = \alpha \mathscr{L}\{x(t)\} = \alpha X(s) \tag{8}$$

$$\mathscr{L}\{\alpha x(t) + \beta y(t)\} = \alpha X(s) + \beta Y(s) \tag{9}$$

Time shift

$$x(t-\tau) = X(s)e^{-s\tau} \tag{10}$$

Initial value theorem

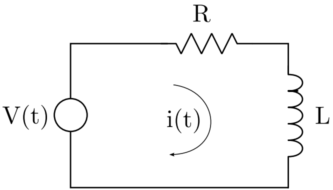
$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) \tag{11}$$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{12}$$

Laplace transform

Consider all initial conditions to be zero.



$$\mathscr{L}\{v(t)\} = \mathscr{L}\left\{Ri(t) + L\frac{di(t)}{dt}\right\} \quad (13)$$

$$V(s) = \quad (14)$$

$V(s)$ is called the forcing function: a term that is only a function of time.

Common forcing signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

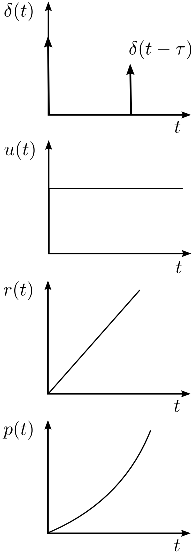
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Partial fraction decomposition

Consider the function $F(s)$ given by

$$F(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{p=0}^{p} a_p s^p} \tag{15}$$

with A and B being polynomials **and** $\mathbf{p} > \mathbf{k}$.

How to split up a complicated fraction into known forms such as:

$$F(s) = \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \dots \frac{a_p}{s+c_p} \ ?$$

Such that:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \dots + \frac{c_p}{s+a_p}\right\}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{c_1}{s+a_1}\right\} + \mathscr{L}^{-1}\left\{\frac{c_2}{s+a_2}\right\} \dots + \mathscr{L}^{-1}\left\{\frac{c_p}{s+a_p}\right\}$$

Partial fraction decomposition

For each factor in the denominator, the term in the decomposition is:

Factor in denominator	Term in partial decomposition
$as + b$	$\frac{c}{as+b}$
$(as + b)^k$	$\frac{c_1}{as+b} + \frac{c_2}{(as+b)^2} + \dots \frac{c_k}{(as+b)^k}$
$as^2 + bx + d$	$\frac{c_1 s+c_2}{as^2+sb+d}$
$(as^2 + bx + d)^k$	$\frac{c_1 s+e_1}{as^2+sb+d} + \frac{c_2 s+e_2}{(as^2+sb+d)^2} + \dots \frac{c_k s+e_k}{(as^2+sb+d)^k}$

Partial fraction decomposition

$$\frac{3s+11}{(s+3)(s+2)}$$

Partial fraction decomposition

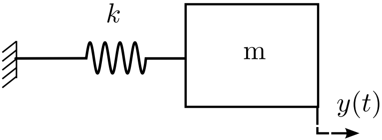
$$\frac{s^2+15}{(s+3)^2(s^2+3)}$$

Laplace transformation

$F(s)$	$f(t)$
1	Unit impulse $\delta(t)$
$\frac{1}{s}$	Unit step function $u(t)$
$\frac{1}{s^2}$	Unit ramp t
$\frac{n!}{s^{n+1}}$	t^n with $n \in \mathbb{N}^+$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{s}{(s+a)(s+b)}$	$\frac{1}{b-a}\left(be^{-bt} - ae^{-at}\right)$

Exercise 11

If the mass is released from rest when the spring is stretched by $y(0) = \alpha$, calculate its position $y(t)$ as a function of time.



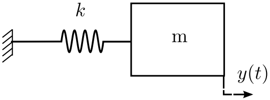
Procedure:

$\rightarrow$ Find the differential equation

$\rightarrow$ Calculate the Laplace transform

$\rightarrow$ Determine the temporal response using the inverse transformation

Exercise 11 - continued



Given $y(0) = \alpha$, $\dot{y}(0) = 0$. Determine $y(t)$.

Exercise 12

Consider the following differential equation in the frequency domain

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

Determine:

$\rightarrow$ The final value of the function $f(t)$ when $t \rightarrow \infty$

$\rightarrow$ The function $f(t)$

Exercise 12 - continued

Final value

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$\lim_{t \to \infty} f(t) = \lim_{s \to 0}[sF(s)]$

Exercise 12 - continued

Final value

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

Exercise 12 - continued

Partial fraction expansion

$$F(s) = \frac{10}{s(s+1)(s+10)} =$$

Exercise 12 - continued

Inverse transformation

$$F(s) = \frac{1}{s} - \frac{10}{9}\frac{1}{s+1} + \frac{1}{9}\frac{1}{s+10}$$

Exercise 13

Consider the following differential equation in the frequency domain

$$F(s) = \frac{1}{s(s+2)^2}$$

Determine:

$\rightarrow$ The final value of the function $f(t)$ when $t \rightarrow \infty$

$\rightarrow$ The function $f(t)$

Exercise 13 - continued

Final value

$$F(s) = \frac{1}{s(s+2)^2}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

Exercise 13 - continued

Inverse transformation

$$F(s) = \frac{1}{4s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

Exercise 14

Assuming all initial conditions are zero, determine the solution of the following differential equation, where the forcing term is $f(t) = 2e^{-t}$.

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = f(t)$$

Exercise 14 - continued

Exercise 14 - continued

$$Y(s) = \frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{2}{s+1} - \frac{2}{s+2}$$

Exercise 15

A laser printer uses a laser beam to print copy rapidly for a computer. The laser is positioned by a control input $r(t)$ so that we have:

$$Y(s) = \frac{6(s+50)}{s^2+40s+300}R(s).$$

The input $r(t)$ represents the desired position of the laser beam. If $r(t) = 1$, determine:

→ The output $y(t)$

→ The final value of $y(t)$

Exercise 15 - continued

1 - Partial fraction expansion

$$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300}R(s).$$

2 - Inverse transform

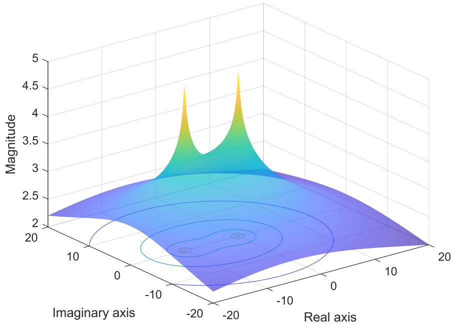
Exercise 15 - continued

2 - Final value of $y(t)$

$$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300} R(s).$$

Skills check 3 - From 2018 assignment 1

The magnitude of the Laplace transform of the temporal response of a unknown mechanical system to an impulse excitation is shown in the figure for a given range of the Laplace variable $s = \sigma + j\omega$.



Based on the graph, indicate whether the following statements about the unknown system are true or false. Justify your answer with a rigorous analysis using the properties of the Laplace transform.

Skills check 3 - continued

(a) If subjected to an impulse-type excitation, the temporal response of the system will exhibit both exponential and sinusoidal components.

(b) For an impulse-type excitation, the final value of the signal when $t \to \infty$ is zero.

(c) The Laplace transform (or the transfer function) of the time response has two poles and no zeros.

(d) The Laplace transform of the system can be approximated by the function

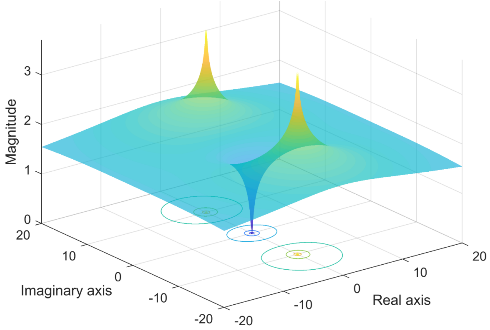
$$H(s) = \frac{a}{s^2 + 10}$$

where $a > 0$ is a constant.[1]

[1] (a) false, (b) false, (c) true (d), false

Skills check 4 - From 2018 midterm examination

The magnitude of the Laplace transform of the signal $x(t)$ is shown in the figure for a given range of the Laplace variable $s = \sigma + j\omega$. Find the approximate function $x(t)$.[2]



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[2] $x(t) = \cos(10t)$

Next class...

- Transfer functions