MECE 3350U
Control Systems

Lecture 20
Stability Margins

Videos in this lecture

Lecture: https://youtu.be/IXSLZ-B0Zn0

Exercise 117: https://youtu.be/HUtiPTA8Mq0

Exercise 118: https://youtu.be/cpr9s0k1aMQ

Exercise 119: https://youtu.be/96mQDIujFEU

Exercise 120: https://youtu.be/PDJ-VSWXfng

Exercise 121: https://youtu.be/_j88ycAA0Fs

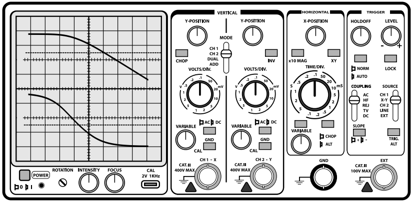
Outline of Lecture 20

By the end of today's lecture you should be able to

- Calculate the gain and phase margin of a system
- Obtain the gain and phase margin from a Bode plot
- Quantify the stability of an open-loop transfer function

Applications

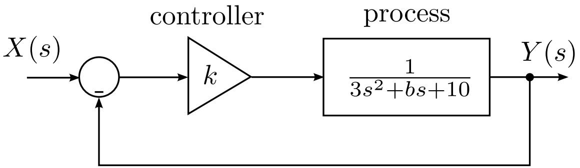
We wish to develop a closed-loop controller for a system whose dynamics is unknown. The frequency response of the open-loop system has been obtained experimentally using an oscilloscope.



What does it tell us about its closed-loop stability?

Applications

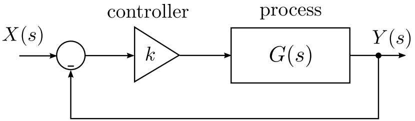
The controller gain k has been specified to the process shown.



If b changes during operation, how can we ensure that the system remains stable?

Bode *vs* Nyquist plots

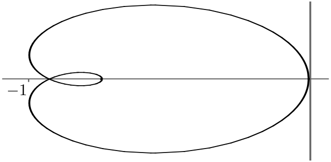
The closed loop system



$$T(s) = \frac{kG(s)}{1 + kG(s)}$$

might be stable for only a range of values of k.

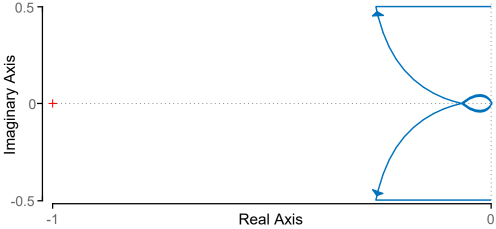
The proximity of the $L(j\omega)$ locus to $-1 + j0$ is a measure of the relative stability of the system.



Bode *vs* Nyquist plot

Consider the open-loop transfer function

$$L(j\omega) = \frac{k}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$



As k is increased, the Nyquist plot approaches $-1 + 0j$ and eventually encircles the 1 point.

The point $-1 + 0j$ can also be expressed in polar form as $1\angle - 180^\circ$

Bode *vs* Nyquist plot

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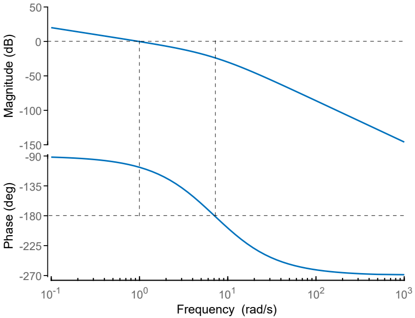
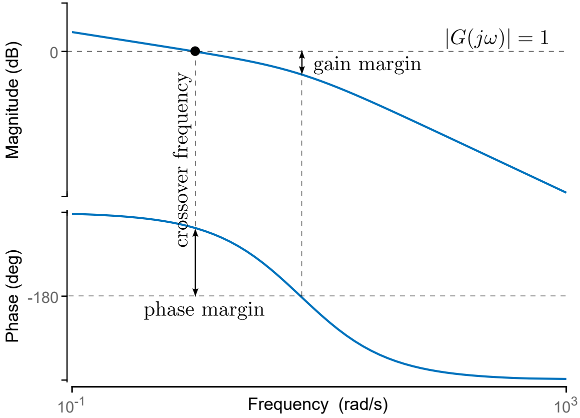


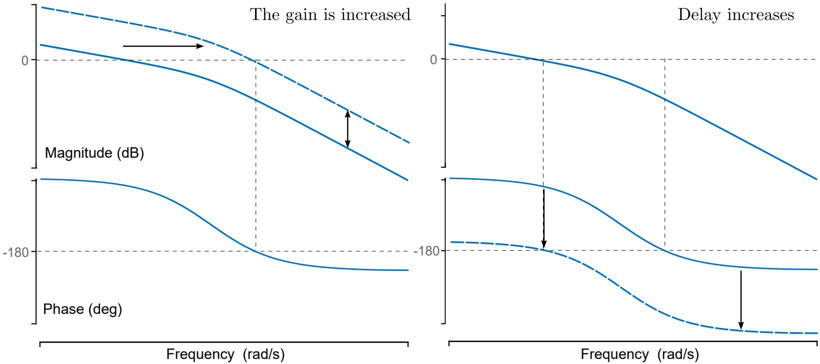
Gain margin: The increase in the loop gain when $\phi = -180^\circ$ that results in $|L(j\omega)| = 1$ or 0 dB.

Phase margin: The amount of phase shift at the crossover frequency that results in $\angle L(j\omega) = -180^\circ$.

Gain and phase margins



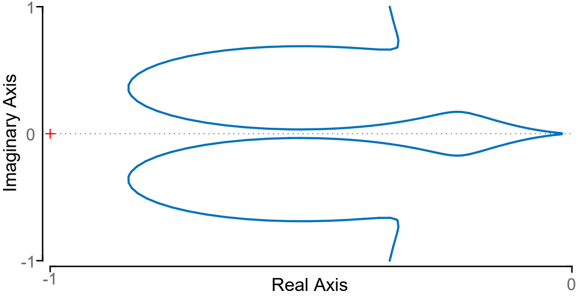
Gain and phase margins



True or false?

The following open-loop transfer function is closed-loop stable for any $k > 0$.

$$L(s) = k\frac{s^2 + 0.1s + 0.5}{s(s+1)(s^2 + 0.05 + 0.5)}$$



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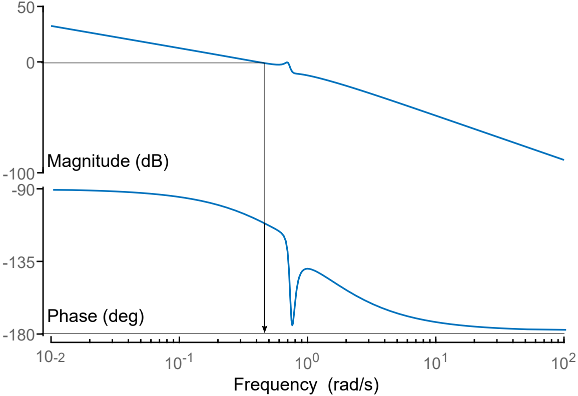


Phase margin

As an example, consider the open-loop second-order system

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \rightarrow \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)} \tag{1}$$

Step 1 - Find the crossover frequency (0 dB)

At the crossover frequency $\omega = \omega_c$, the magnitude is 1. Find ω_c that gives

$$\frac{\omega_n^2}{\omega_c\sqrt{\omega_c^2 + 4\zeta^2\omega_n^2}} = 1.$$

Step 2 - Find the phase of $G(j\omega)$ at ω_c for ω_c found in Step 1, i.e. $\angle G(j\omega_c)$

$$\phi = -90^\circ - \tan\left(\frac{\omega_c}{2\zeta\omega_n}\right)$$

Step 3 - The margin phase is $PM = 180 - |\phi|$

If $PM < 0$, the system is closed-loop unstable.

Gain margin

Consider the same open-loop second-order system

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \rightarrow \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}$$

Step 1 - Find the frequency ω_f where $\angle|G(j\omega)| = -180^\circ$

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)} = \frac{\omega_n^2}{-\omega_f^2 + j2\zeta\omega_n\omega_f} \times \frac{-\omega_f^2 - j2\zeta\omega_n\omega_f}{-\omega_f^2 - j2\zeta\omega_n\omega_f}$$

$$G(j\omega) = -\frac{\omega_n^2\omega_f^2}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2} - j\frac{2\zeta\omega_n^3\omega_f}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2}$$

At ω_f, $\Im[G(j\omega_f)] = 0$ (imaginary part is zero)

$$-\frac{2\zeta\omega_n^3\omega_f}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2} = 0$$

$\omega_f = 0$ Not a valid frequency

$\omega_f = \infty$ What does it mean?

$\omega_f =$ constant. Proceed to Step 2

Gain margin

Step 2 - Find the gain of $G(j\omega)$ at $\omega = \omega_c$, i.e., $|G(j\omega_f)| = k_{MG}$

$$k_{MG} = \frac{\omega_n^2}{\omega_f\sqrt{\omega_f^2 + 4\zeta^2\omega_n^2}}$$

Then gain margin in Decibels is

$$MG = -20\log(k_{MG})$$

$\rightarrow MG > 0$: Stable. The gain can be multiplied by k_{MG} **dB** before the system becomes marginally stable (or MG dB can be added before instability);

$\rightarrow MG = 0$ The system is marginally stable.

$\rightarrow MG < 0$: Unstable. The gain can be divided by k_{MG} dB before the system becomes marginally stable (or MG dB must be subtracted to achieve stability).

Exercise 117

A unit feedback control system has a loop transfer function

$$L(s) = \frac{k}{s(s+2)(s+10)}$$

For $k = 50$, determine the cross over frequency, the gain margin, and the phase margin.

Exercise 117 - continued

$$L(s) = \frac{k}{s(s+2)(s+10)}$$

Exercise 117 - continued

$$L(s) = \frac{k}{s(s+2)(s+10)}$$

Exercise 118

A unit feedback control system has a loop transfer function

$$L(s) = \frac{k}{(s+1)^2}$$

Determine the gain k so that the phase margin is 60°

Exercise 118 - continued

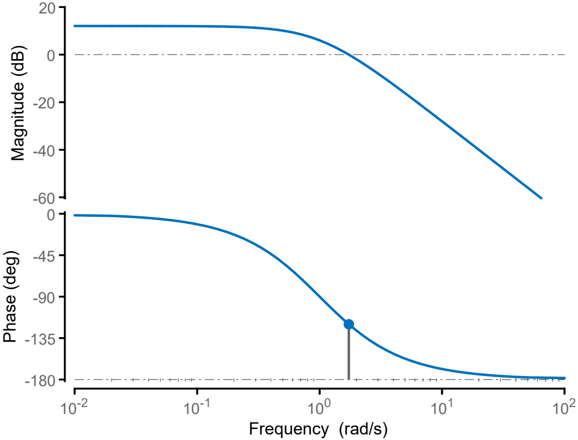
$$L(s) = \frac{k}{(s+1)^2}$$

Exercise 118 - continued

$$L(s) = \frac{k}{(s+1)^2}$$

Exercise 118 - continued

Bode plot for $k = 4$.



Exercise 119

A system has a loop transfer function

$$T(s) = 10.5\frac{1 + s/5}{s(1 + s/2)(1 + s/10)}$$

Show that the crossover frequency is 5 rad/s and that the phase margin is 40°

Exercise 119 - continued

$$T(s) = 10.5 \frac{1 + s/5}{s(1 + s/2)(1 + s/10)}$$

Exercise 119 - continued

$$T(s) = 10.5 \frac{1 + s/5}{s(1 + s/2)(1 + s/10)}$$

Exercise 120 - continued

Consider a unit feedback system with the loop transfer function

$$L(s) = \frac{k}{s(s+1)(s+4)}$$

(a) For $k = 5$, show that the gain margin is 12 dB

(b) If we wish to achieve a gain margin of 20 dB, determine the value of k

Exercise 120 - continued

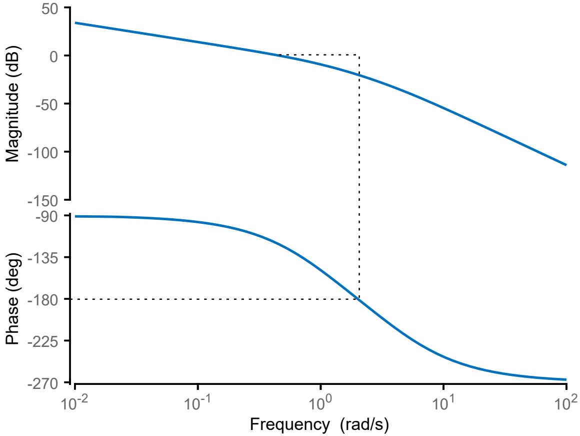
$$L(s) = \frac{k}{s(s+1)(s+4)}$$

Exercise 120 - continued

$$L(s) = \frac{k}{s(s+1)(s+4)}$$

Exercise 120 - continued

Bode diagram for $k = 2$.



Exercise 121 - Using Matlab

Consider a unit feedback system with a proportional controller such that the loop transfer function is

$$L(s) = k\frac{s^2 + 0.1s + 0.5}{s(s+1)(s+2)(s^2 + 0.05s + 0.5)}$$

with

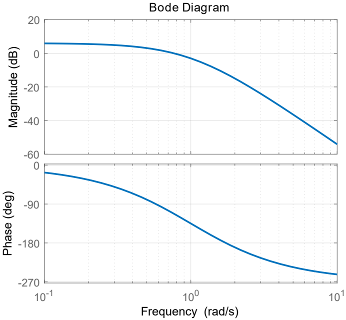
Using Matlab, plot the phase and gain margins for $0 \leq k \leq 10$. Specify the maximum value of k that results in a stable closed loop system.

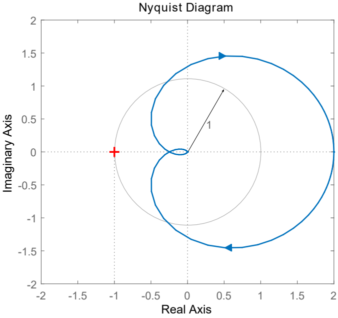
Matlab code for Exercise 121

```
clear all; close all;
s = tf([1 0],[1]);
i = 1;
for k = 1:0.05:10;
H = k*(s^2 + 0.1*s + 0.5)/(s*(s + 1)*(s^2 + 0.05*s + 0.5)*(s + 2));
[Gm,Pm,Wcg,Wcp] = margin(H);
PhaseM(i) = Pm;
GainM(i) = Gm;
K(i) = k;
i = i+1;
end
yyaxis left
plot(K, PhaseM);
yyaxis right
plot(K, GainM);
yyaxis left
title('Phase and gain margins')
xlabel('k')
ylabel('phase margin [deg]')
yyaxis right
ylabel('Gain Margin [dB]')
```

Skills check 50 - From 2018 final examination

The Bode and Nyquist plots of a given transfer function $G(s)$ are shown below. Identify the gain and phase margins of $G(s)$ on each diagram (*3 marks*):





Skills check 51 - From 2018 final examination

Consider the following transfer function:

$$G(s) = \frac{200}{(s+1)(s^2+5s+100)}$$

(a) Calculate the phase and magnitude of $G(s)$ at 1 rad/s.

(b) Calculate the phase margin.

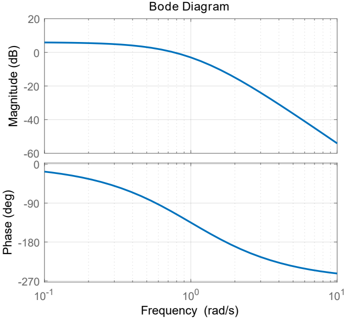
(c) Draw the Bode plot of $G(s)$. Based on the Bode plot, specify the approximate gain at the cut-off frequencies and at 10^3 rad/s .

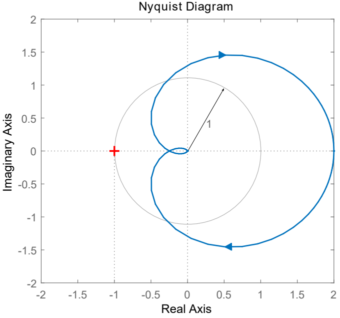
(d) Calculate the gain margin (*4 marks*).

Answers: (a) 3dB, -50°, (b) 144°, (d) 8.87 dB

Skills check 52 - From 2018 final examination

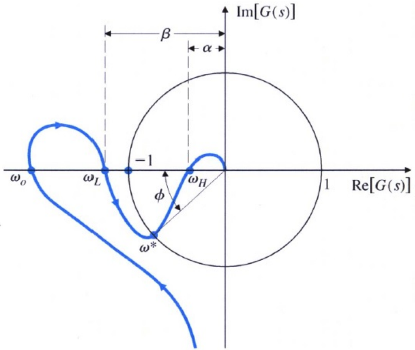
The Bode and Nyquist plots of a given transfer function $G(s)$ are shown below. Identify the gain and phase margins of $G(s)$ on each diagram (*3 marks*):





Skills check 53 - From 2018 deferred final examination

The Nyquist plot for a control system resembles the one shown below. What are the gain and phase margins ? (*3 marks*).



Answers: GM $= 1/\alpha$, and $1/\beta$. P.M $= \phi$

Student course feedback survey

https://www.cci-survey.ca/ontariotechu/ca/

Next class...

- Space state models