MECE 3350U
Control Systems

Lecture 17

Bode Plots 2/2

Videos in this lecture

Lecture: https://youtu.be/miRXSjKEV-Y

Bode plots explained: https://youtu.be/uyYwNWG1Kuc

Exercise 100: https://youtu.be/tLvQgQBVpBI

Exercise 101: https://youtu.be/IvHcw4Qcuiw

Exercise 102: https://youtu.be/04MAgVDMg08

Exercise 103: https://youtu.be/pdBAqQ7wSns

Exercise 104: https://youtu.be/gJ_ae0hj1AQ

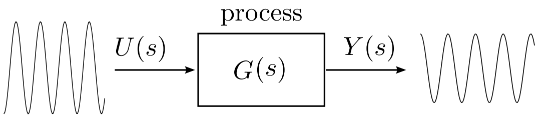
Outline of Lecture 17

By the end of today's lecture you should be able to

- Represent magnitude and phase in a Bode plot
- Draw the Bode plot for functions having complex poles

Frequency response

Frequency response: The steady-state response of the system to a sinusoidal input signal.



$$y(t) = AM\sin(\omega_0 t + \phi) \tag{1}$$

where A the amplitude of the input signal and

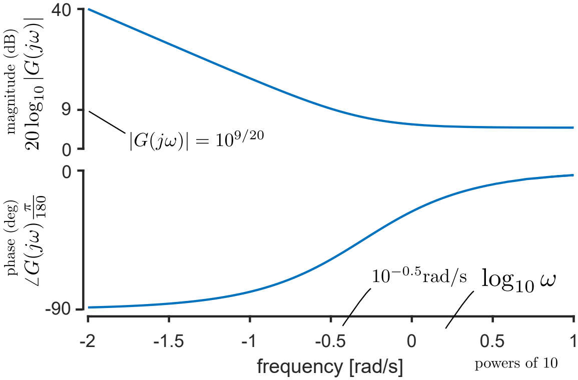
$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0} = \sqrt{[\Re G(j\omega_0)]^2 + [\Im G(j\omega_0)]^2}$$

$$\angle G(j\omega_0) = \phi = \tan^{-1}\left(\frac{\Im[G(j\omega_0)]}{\Re[G(j\omega_0)]}\right)$$

Bode plots

The vertical axis shows the phase ϕ and gain $20\log(G(j\omega))$

The horizontal axis is logarithmic $\log_{10}(\omega)$



Bode plots - review

Given a transfer function

$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{2}$$

The gain is

$$[G(j\omega)| = \ 20\log\left[k\frac{\prod_{i=1}^{n}(j\omega+z_i)}{\prod_{i=1}^{m}(j\omega+p_i)}\right] \tag{3}$$

Since $\log(a \times b) = \log(a) + \log(b)$, we can rewrite the gain as

$$[G(j\omega)| = 20\log(k) + \sum_{i=1}^{n}[20\log(j\omega+z_i)] + \sum_{i=1}^{m}\left[20\log\frac{1}{(j\omega+p_i)}\right] \tag{4}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.

Bode plots - review

Given a transfer function

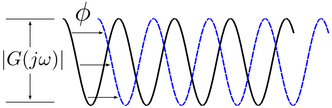
$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{5}$$

The phase is

$$\angle[G(j\omega)| = \phi = \tan^{-1}\left[\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]}\right] \tag{6}$$

$$\phi = \angle(k) + \sum_{i=1}^{n}[\angle(j\omega+z_i)] + \sum_{i=1}^{m}\left[\angle\frac{1}{j\omega+p_i}\right] \tag{7}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.



Bode plot building blocks - review

1 - Constant gain

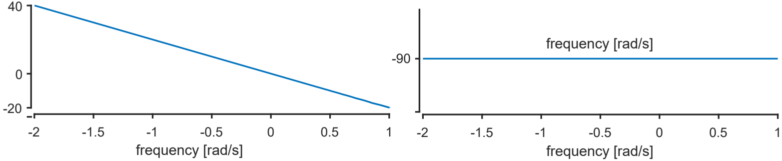
→ Gain: $|k|$ or $20\log(|k|)$

→ Phase: $\phi = 0 \; \forall \; \omega$ if $k > 0$, -180° otherwise

2 - Pole at the origin

→ Gain: $-20\log(\omega)$

→ Phase: $-90^\circ \; \forall \; \omega$

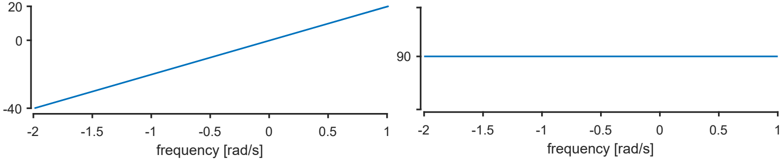


Bode plot building blocks - review

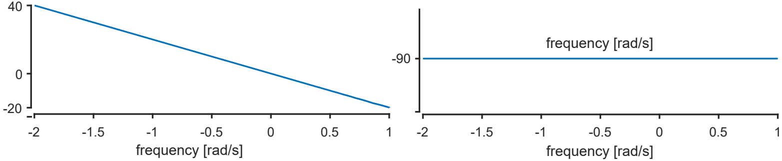
3 - Zero at the origin

→ Gain: $20\log(\omega)$

→ Phase: $90^\circ\ \forall\ \omega$



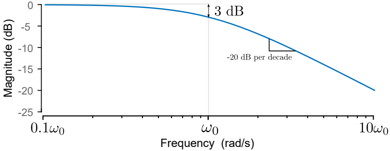
Note that this is the negative of a pole at the origin:

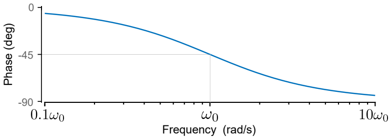


Bode plot building blocks - review

4 - Real pole: $G(s) = \frac{1}{\frac{s}{\omega_0}+1}$, $\omega_0 \in \Re^*$

	$\omega << \omega_0$	$\omega = \omega_0$	$\omega >> \omega_0$
Gain	0	-3 dB	$-20\log\frac{\omega}{\omega_0}$
Phase	0	-45°	-90°

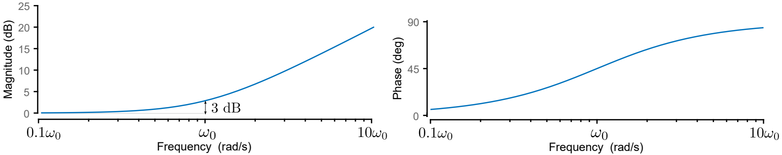




Bode plot building blocks - review

5 - Real zero: $G(s) = \frac{s}{\omega_0} + 1,\ \omega_0 \in \Re^*$

	$\omega << \omega_0$	$\omega = \omega_0$	$\omega >> \omega_0$
Gain	0	+3 dB	$20 \log \frac{\omega}{\omega_0}$
Phase	0	45°	90°



The real zero is the negative of a real pole on the Bode plot

Exercise 100

Draw the approximate Bode plot of the transfer function

$$G(s) = \frac{(s+10)}{(s+1)^2(s+100)}$$

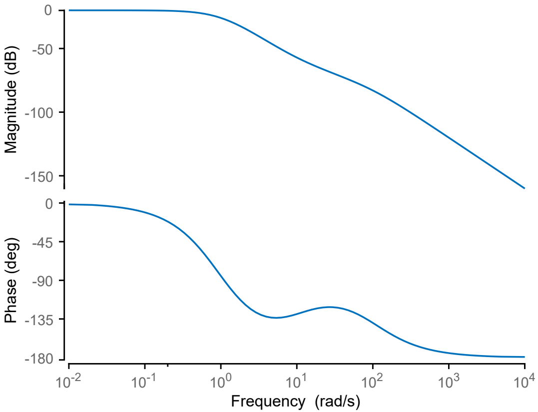
Exercise 100 - continued

Draw the approximate Bode plot of the transfer function

$$G(s) = \frac{(s+10)}{(s+1)^2(s+100)}$$

Exercise 100 - continued

Result using Matlab



Bode plot building blocks

6 - Complex conjugate poles $G(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$

The equation can be rearranged as

$$G(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1} \quad \rightarrow \quad G(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j2\zeta\left(\frac{\omega}{\omega_0}\right)} \tag{8}$$

$$G(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j2\zeta\left(\frac{\omega}{\omega_0}\right)} \times \frac{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 - j2\zeta\left(\frac{\omega}{\omega_0}\right)}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 - j2\zeta\left(\frac{\omega}{\omega_0}\right)} \tag{9}$$

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} \tag{10}$$

Bode plot building blocks

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

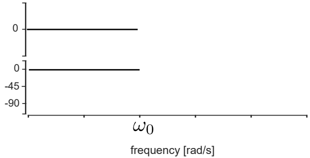
Case 1: $\omega << \omega_0$,

$\rightarrow$ Thus: $\omega/\omega_0 \approx 0$ and $G(j\omega)$ simplifies to

$$G(j\omega) \approx 1 + 0j$$

$\rightarrow$ The gain is $20\log(\sqrt{1^2+0^0}) = 0$ dB

$\rightarrow$ The phase is $\phi = \tan^{-1}(\frac{0}{1}) = 0^\circ$



Bode plot building blocks

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

Case 2: $\omega >> \omega_0$

$\rightarrow$ Thus $G(j\omega)$ simplifies to

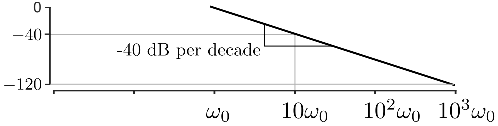
$$G(j\omega) \approx -\frac{1}{\left(\frac{\omega}{\omega_n}\right)^2} - j\frac{2\zeta}{\left(\frac{\omega}{\omega_0}\right)^3}$$

$\rightarrow$ The gain is

$$G = 20\log\sqrt{\left[-\left(\frac{\omega}{\omega_n}\right)^{-2}\right]^2 + \left[-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}\right]^2} \approx 20\log\left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$\boxed{G = -40\log\left(\frac{\omega}{\omega_0}\right)}$$

Bode plot building blocks

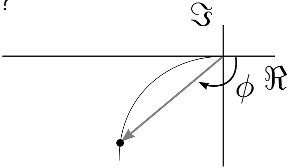


Still when $\omega >> \omega_0$, let us look at the phase:

$$G(j\omega) \approx -\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2} - j\frac{2\zeta}{\left(\frac{\omega}{\omega_0}\right)^3}$$

$$\phi = \tan^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}}{-\left(\frac{\omega}{\omega_0}\right)^{-2}}\right] = \tan^{-1}\underbrace{\left[2\zeta\frac{\omega_0}{\omega}\right]}_{\rightarrow 0} = -180^\circ$$

Why -180° instead of 0°?



Bode plot building blocks

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

Case 3: $\omega = \omega_0$

$\rightarrow$ Thus $G(j\omega)$ simplifies to

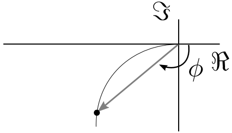
$$G(j\omega) = 0 - j\left(\frac{1}{2\zeta}\right)$$

$\rightarrow$ The gain is

$$G = 20\log\sqrt{\left(\frac{-1}{2\zeta}\right)^2} = 20\log(2\zeta)^{-1} = -20\log(2\zeta)$$

$\zeta = 0.5,\ G = 0$ dB
$\zeta < 0.5$, there is a peak at $\omega = \omega_0$
$\zeta > 0.5$, there is a negative gain at $\omega = \omega_0$
$\zeta = 0$, $\mathrm{G}(\omega = \omega_0) \rightarrow \infty$

Bode plot building blocks



When $\omega = \omega_0$, the phase is

$$G(j\omega) = 0 - j\left(\frac{1}{2\zeta}\right)$$

$$\phi = \tan^{-1}\left(-\frac{\frac{1}{2\zeta}}{c}\right)_{c\to 0} = -90^\circ$$

In summary

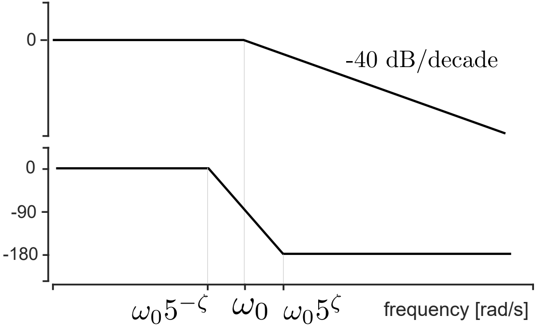
	$\omega << \omega_0$	$\omega >> \omega_0$	$\omega = \omega_0$
Gain	0	-40 dB/decade	$-20\log(2\zeta)$
Phase	0°	-180°	-90°

Notice that this analogous to having two equal real poles.

Bode plot building blocks

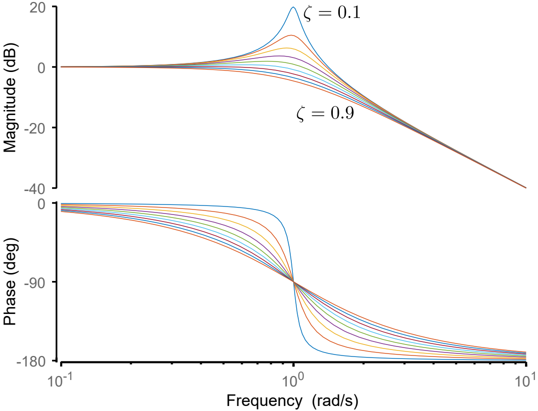
Summary - Bode plots for complex poles

	$\omega << \omega_0$	$\omega >> \omega_0$	$\omega = \omega_0$
Gain	0	-40 dB/decade	$-20\log(2\zeta)$
Phase	0°	-180°	-90°



Influence of the damping ratio

$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}$$



Resonance frequency

The frequency at which the gain reaches its maximum value is called the **resonance frequency**.

The resonance frequency satisfies $\omega = \omega_r$

$$\frac{\partial}{\partial \omega}\sqrt{\left(\frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}\right)^2+\left(\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}\right)^2}=0$$

$$\omega_r = \omega_0\sqrt{1-2\zeta^2}, \text{ for } \zeta < \frac{\sqrt{2}}{2} \tag{11}$$

Thus, the maximum value M_ω of $|G(j\omega)|$ is

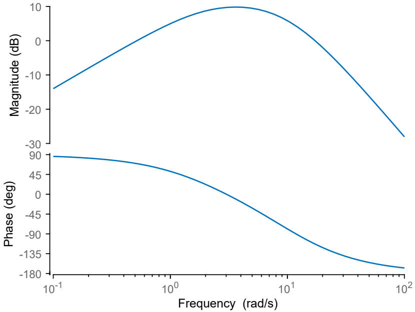
$$M_\omega = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \text{ for } \zeta < \frac{\sqrt{2}}{2} \tag{12}$$

Exercise 101

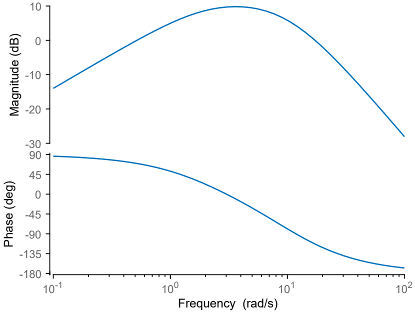
The frequency response of a dynamic system has many practical applications and is often used in order to estimate the system parameters. Knowing that a system transfer function is

$$G(s) = k\frac{s}{(s+a)(s^2+20s+100)}$$

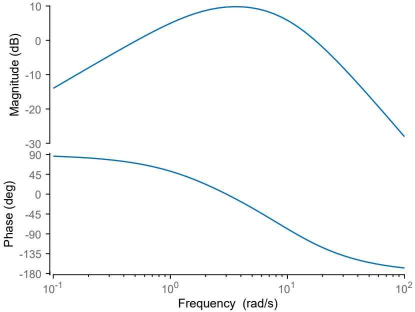
And its frequency response is shown in the Bode plot, determine k and a.



Exercise 101 - continued



Exercise 101 - continued



Exercise 102

A low-pass filter is a filter that passes signals with a frequency lower than a certain cut-off frequency and attenuates signals with frequencies higher than the cut-off frequency. A hypothetical filter has the transfer function

$$G(s) = \frac{4}{s^2 + 0.4s + 4}$$

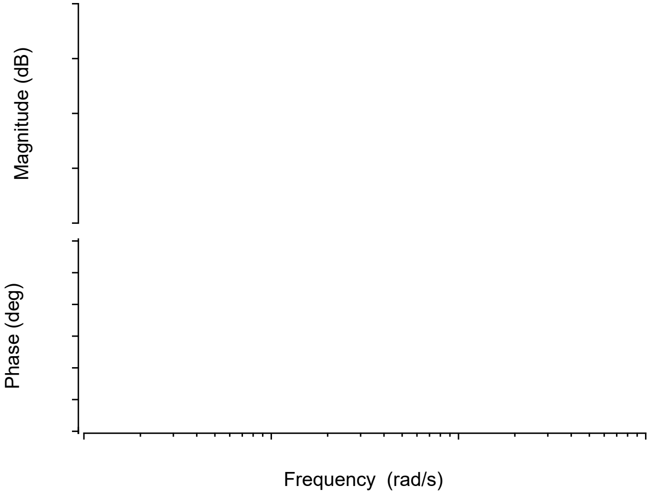
Sketch its frequency response.

Exercise 102 - continued

$$G(s) = \frac{4}{s^2 + 0.4s + 4}$$

Exercise 102 - continued

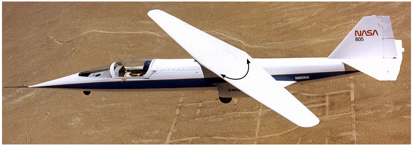
$\omega_0 = 1$ rad/s and $\omega_0 = 2$ rad/s (complex poles).



Exercise 103

The experimental oblique wing aircraft has a wing that pivots. Its control system loop transfer function is

$$G(s) = \frac{4(0.5s+1)}{(2s+1)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$

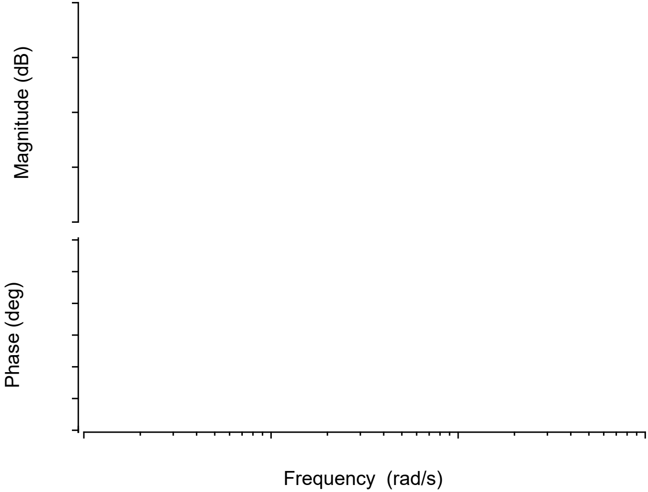


Sketch its frequency response.

Exercise 103 - continued

$$G(s) = \frac{4(0.5s+1)}{(2s+1)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$

Exercise 103 - continued



Exercise 104 - Matlab problem

Consider the closed-loop transfer function

$$R(s) = \frac{30}{s^2 + s + 30}$$

Develop a Matlab code to obtain the Bode plot and verity that the resonant frequency is 5.44 rad/s and that the peak magnitude is 14.8 dB.

Compare the results of you code with the results of "Bode(R)" function.

Skills check 40 - From 2018 final examination

Question 6) Radio telescopes are used to study the radio frequency portion of the electromagnetic spectrum emitted by astronomical objects. One critical problem of driving large telescopes is the form of the system transfer function that has a structural resonance. A large telescope with a diameter of 20 meters, for example, is subject to large wind gust torques that can affect positioning accuracy. Consider an antenna, drive motor, and amplifier system with the following transfer function

$$G(s) = \frac{200}{(s+1)(s^2+5s+100)}$$

(a) Calculate the phase and magnitude of $G(s)$ at 1 rad/s.

(b) Draw the Bode plot of $G(s)$. Based on the Bode plot, specify the approximate gain at 1 rad/s.

Answer using Matlab: bode(tf([150],[1 18 55 150]))

Skills check 41 - From 2018 deferred final examination

Question 6) Radio telescopes are used to study the radio frequency portion of the electromagnetic spectrum emitted by astronomical objects. One critical problem of driving large telescopes is the form of the system transfer function that has a structural resonance. A large telescope with a diameter of 20 meters, for example, is subject to large wind gust torques that can affect positioning accuracy. Consider an antenna, drive motor, and amplifier system with the following transfer function

$$G(s) = 150 \frac{s}{(s+0.7)^2(s^2+s+49)}$$

(a) Calculate the phase and magnitude of $G(s)$ at 10^{-3}, 1, and 10^3 rad/s.

(b) Draw the Bode plot of $G(s)$.

Answer using Matlab: bode(tf([150 0],[1 2.4 50.89 69.09 24.01]))

Skills check 42

Consider an unit feedback system whose open-loop transfer function is

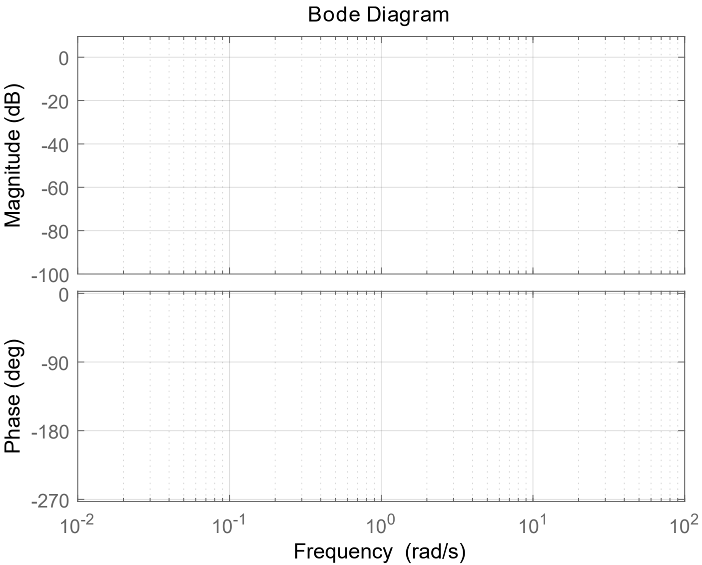
$$G(s) = \frac{k}{(s+1)(s^2+4s+25)}$$

(a) Draw Bode plot of the open-loop system for $k = 75$.

(b) Calculate the phase and magnitude of $G(s)$ at 1 rad/s for $k = 75$.

Answer using Matlab: bode(tf([75],[1 5 29 25]))

Skills check 42 - continued



Skills check 43

Consider an unit feedback system whose open-loop transfer function is

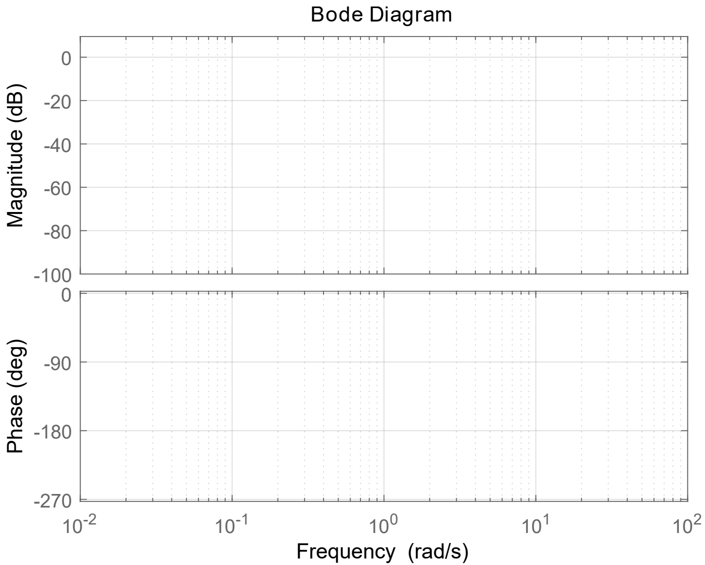
$$G(s) = \frac{s}{(s+1)(s^2+4s+25)}$$

(a) Draw Bode plot of the open-loop system.

(b) Calculate the phase and magnitude of $G(s)$ at 1 rad/s for $k = 75$.

Answer using Matlab: bode(tf([75],[1 5 29 25]))

Skills check 43 - continued



Skills check 44

Consider an unit feedback system whose open-loop transfer function is

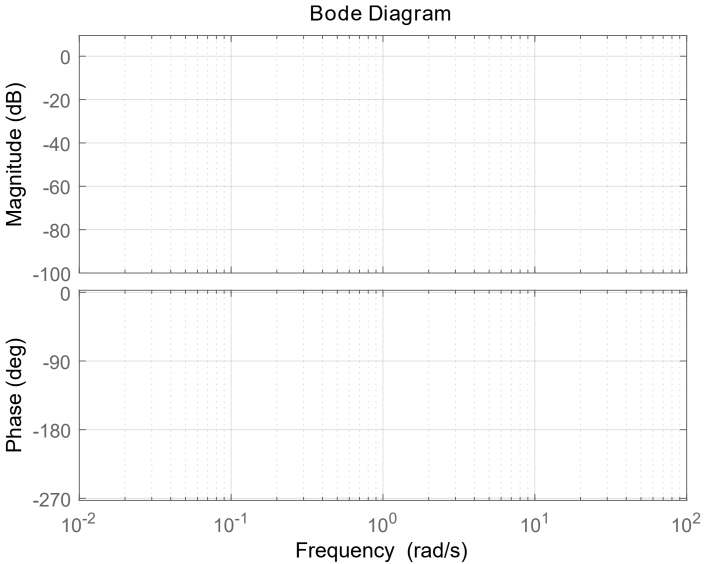
$$G(s) = \frac{25}{s(s^2 + 4s + 25)}$$

(a) Draw Bode plot.

(b) Calculate the phase and magnitude of $G(s)$ at 1 rad/s for $k = 75$.

Answer using Matlab: bode(tf([25],[1 4 25 0]))

Skills check 44 - continued



Next class...

- Stability in the frequency domain