

MECE 3350U  
Control Systems

Lecture 16  
Bode Plots 1/2

## Videos in this lecture

Lecture: <https://youtu.be/HZH9mFtfgnI>

Exercise 93: <https://youtu.be/-0goGrVeqHQ>

Exercise 94: <https://youtu.be/ShypfZHBPTA>

Exercise 95: <https://youtu.be/9fz6evCg4iw>

Exercise 96: <https://youtu.be/iYiK-mTD65E>

Exercise 97: <https://youtu.be/7HpM6zzBzcs>

Exercise 98: [https://youtu.be/\\_Fgnzzq6\\_3Q](https://youtu.be/_Fgnzzq6_3Q)

Exercise 99: <https://youtu.be/ijSaFSJdhiI>

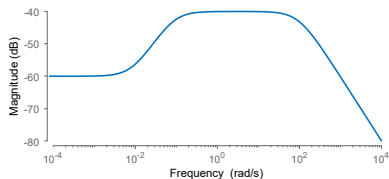
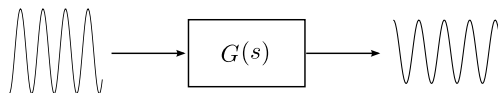
## Outline of Lecture 16

By the end of today's lecture you should be able to

- Understand the concept of frequency response
- Determine the magnitude of a transfer function
- Determine the phase of a transfer function
- Represent magnitude and phase in a Bode plot

## Applications

The response of a unknown dynamic system to a sinusoidal excitation of variable frequency was measured using an oscilloscope.



What can we infer about the system ?

## Applications

An object free to vibrate tends to do so at a specific rate called the object's or resonant frequency.

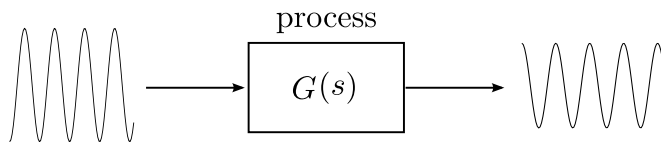


<https://goo.gl/sv2nPQ>

How can we identify the resonance frequency of a system?

## Frequency response

Industrial design of control system is accomplished using frequency-response methods more often than any other.

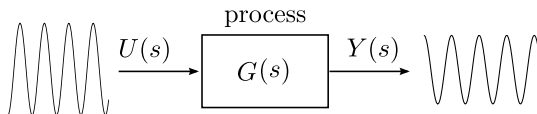


### Advantages:

- Raw measurements of the output are sufficient for design
- Simple to compute
- No need to know the dynamic model of the system

## Frequency response

Frequency response: The steady-state response of the system to a sinusoidal input signal.



Consider a system described by

$$\frac{Y(s)}{U(s)} = G(s) \quad (1)$$

where the input  $u(t)$  is a sine wave with an amplitude  $A$ :

$$u(t) = A \sin(\omega_0 t) \quad \rightarrow \quad U(s) = A \frac{\omega_0}{s^2 + \omega_0^2} \quad (2)$$

With zero initial conditions, we have

$$Y(s) = G(s) A \frac{\omega_0}{s^2 + \omega_0^2} \quad (3)$$

## Frequency response

$$Y(s) = G(s)A \frac{\omega_0}{s^2 + \omega_0^2} \quad (4)$$

Using partial-fraction expansion, the solution takes the form of

$$Y(s) = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \dots + \frac{\alpha_n}{s - p_n} + \frac{\alpha_0}{s + j\omega_0} + \frac{\alpha_0^*}{s - j\omega_0} \quad (5)$$

where  $\alpha_0^*$  is the complex conjugate of  $\alpha_0$ .

The time response is

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \dots + \alpha_n e^{p_n t} + 2|\alpha_0| \sin(\omega_0 t + \phi) \quad (6)$$

with

$$\phi = \tan^{-1} \left( \frac{\Im(\alpha_0)}{\Re(\alpha_0)} \right) \quad (7)$$

## Frequency response

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \dots + \alpha_n e^{p_n t} + 2|\alpha_0| \sin(\omega_0 t + \phi) \quad (8)$$

Provided that  $p_i < 0 \forall i$ , the exponential terms die out eventually and the steady-state response  $y(t \rightarrow \infty)$  is

$$y(t) = 2|\alpha_0| \sin(\omega_0 t + \phi) \quad (9)$$

which can be expressed as

$$y(t) = AM \sin(\omega_0 t + \phi) \quad (10)$$

where

$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0} = \sqrt{[\Re G(j\omega_0)]^2 + [\Im G(j\omega_0)]^2}$$

$$\phi = \tan^{-1} \left( \frac{\Im(G(j\omega_0))}{\Re(G(j\omega_0))} \right) = \angle G(j\omega_0)$$

## Frequency response

$$G(j\omega_0) = R(\omega_0) + jX(\omega_0)$$

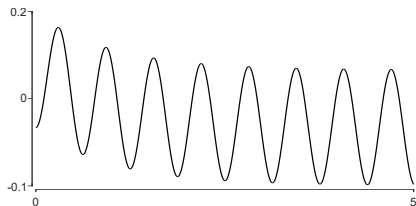
$$M = |G(s)| = \sqrt{R(\omega_0)^2 + X(\omega_0)^2}$$

$$\angle G(j\omega_0) = \tan^{-1} \left( \frac{X(\omega_0)}{R(\omega_0)} \right)$$

Note that in the above,  $s = j\omega$  rather than  $s = \sigma + j\omega$ . Why?

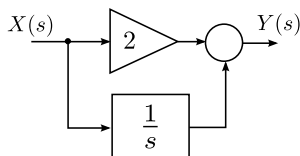
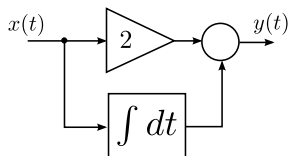
Back to the temporal response, what can we conclude?

$$y(t) = AM \sin(\omega_0 t + \phi)$$



## Example

Consider  $x(t) = \sin(0.5t)$ . What is  $y(t)$ ?



$$y(t) = 2 \sin(0.5t) + \int \sin(0.5t) dt$$

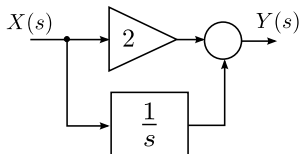
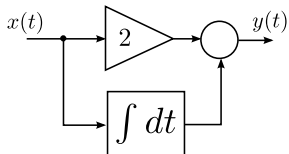
$$y(t) = 2 \sin(0.5t) - 2 \cos(0.5t)$$

Trigonometric identity:  $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \tan^{-1} b/a)$

$$y(t) = 2\sqrt{2} \sin(0.5t - 0.785)$$

## Example

What is the magnitude of  $Y(s)$ ?



$$Y(s) = \left(2 + \frac{1}{s}\right) X(s)$$

$$\omega = 0.5 \text{ rad/s.}$$

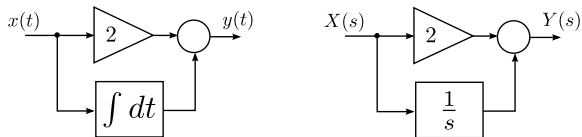
$$|G(s)| = \left(2 + \frac{1}{0.5j}\right) = \left(2 + \frac{1}{0.5j} \times \frac{j}{j}\right)$$

$$|G(s)| = |2 - 2j|$$

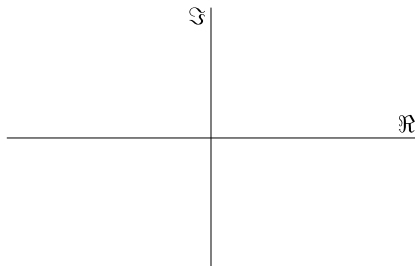
$$|G(s)| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\angle G(s) = \tan^{-1}(-2/2) = 0.785 \text{ (rad)}$$

## Frequency response



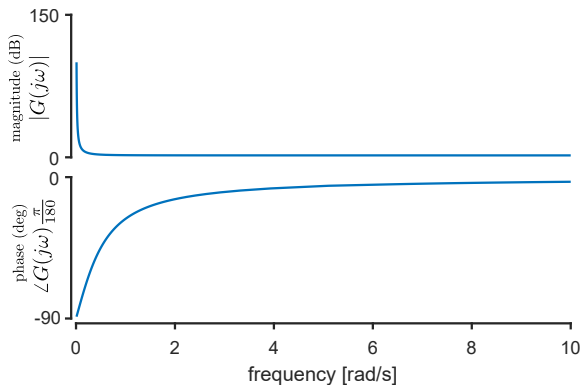
$$G(s) = 2 + \frac{1}{s} \rightarrow G(j\omega) = 2 - \frac{1}{\omega j}$$



## Magnitude and phase plots

Recall that  $G(j\omega) = 2 + \frac{1}{\omega}j$ .

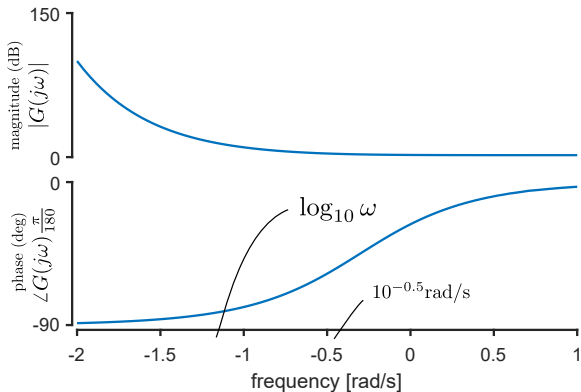
We can plot the magnitude and gain for any given frequency range.



## Magnitude and phase plots

The horizontal axis is typically logarithmic

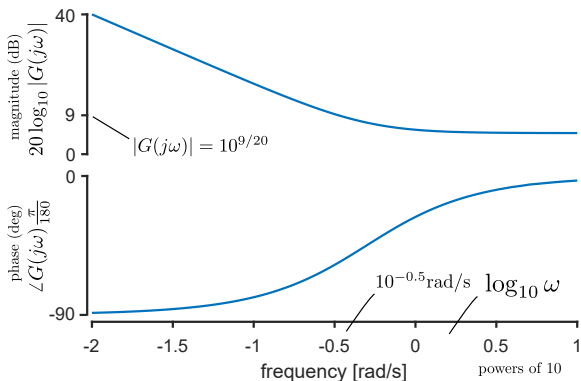
The horizontal axis shows  $\log_{10}(\omega)$



## Bode plot

The magnitude is expressed in Decibels

The magnitude is shown as  $20 \log_{10} |G(j\omega)|$ . This is called the "gain".



## Gain and phase

Given a transfer function

$$G(s) = k \frac{\prod_{i=1}^n (s + z_i)}{\prod_{i=1}^m (s + p_i)} \quad (11)$$

The gain is

$$|G(j\omega)| = 20 \log \left[ k \frac{\prod_{i=1}^n (j\omega + z_i)}{\prod_{i=1}^m (j\omega + p_i)} \right] \quad (12)$$

Since  $\log(a \times b) = \log(a) + \log(b)$ , we can rewrite the gain as

$$|G(j\omega)| = 20 \log(k) + \sum_{i=1}^n [20 \log(j\omega + z_i)] + \sum_{i=1}^m \left[ 20 \log \frac{1}{(j\omega + p_i)} \right] \quad (13)$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of  $G(s)$ .

## Gain and phase

Given a transfer function

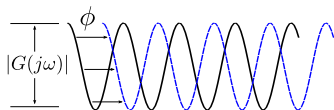
$$G(s) = k \frac{\prod_{i=1}^n (s + z_i)}{\prod_{i=1}^m (s + p_i)} \quad (14)$$

The phase is

$$\angle[G(j\omega)] = \phi = \tan^{-1} \left[ \frac{\Im[G(j\omega)]}{\Re[G(j\omega)]} \right] \quad (15)$$

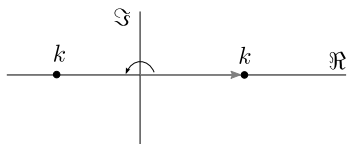
$$\phi = \angle(k) + \sum_{i=1}^n [\angle(j\omega + z_i)] + \sum_{i=1}^m \left[ \angle \frac{1}{j\omega + p_i} \right] \quad (16)$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of  $G(s)$ .



## Bode plot building blocks

**Constant gain**  $G(s) = k$



$$|G(\omega)| = 20 \log |k| \quad \forall \omega$$

$$\phi(\omega) = \tan^{-1} \left( \frac{0}{k} \right) = 0 \quad \forall \omega$$

The gain curve is a horizontal line on the Bode plot.

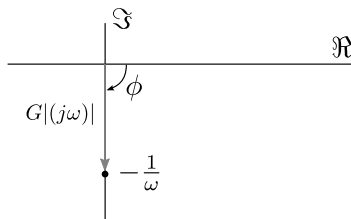
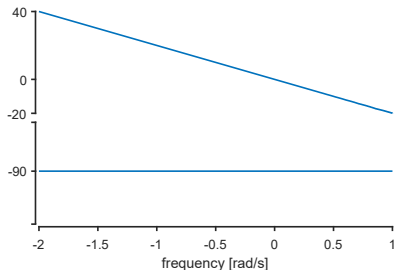
If  $k < 0$ , the magnitude remains  $20 \log |k|$  and the phase becomes  $-180^\circ$ .

## Bode plot building blocks

**Pole at the origin**  $G(s) = \frac{1}{s}$

$$|G(\omega)| = 20 \log \left| \frac{1}{j\omega} \right| = 20 \log(1) - 20 \log \omega = -20 \log \omega$$

$$\phi(\omega) = \tan^{-1} \left( \frac{-1/\omega}{c} \right)_{c \rightarrow 0} = -90^\circ$$

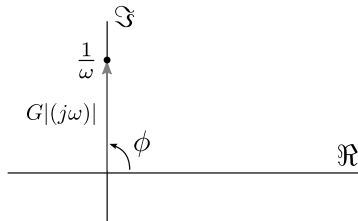
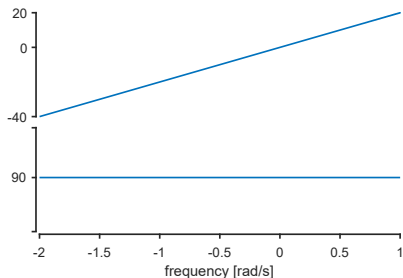


## Bode plot building blocks

**Zero at the origin**  $G(s) = s$

$$|G(\omega)| = 20 \log |j\omega| = 20 \log \omega$$

$$\phi(\omega) = \tan^{-1} \left( \frac{\omega}{c} \right)_{c \rightarrow 0} = 90^\circ$$

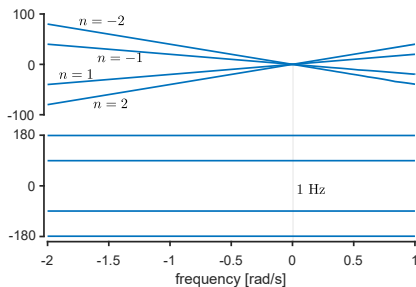


## Bode plot building blocks

**Multiple zeros or poles at the origin**  $G(s) = s^n$ ,  $n \geq 0$  or  $n \leq 0$

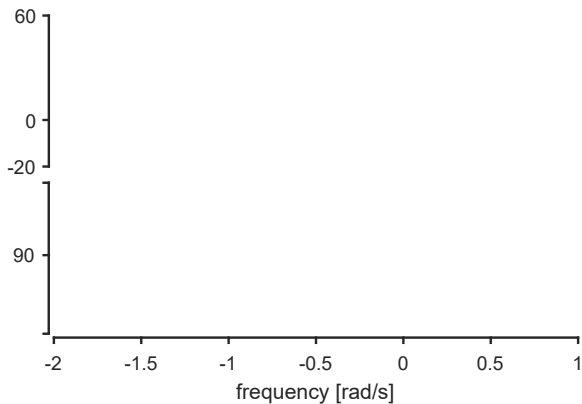
$$|G(\omega)| = 20 \log |(j\omega)^n| = n \times 20 \log \omega$$

$$\phi(\omega) = \tan^{-1} \left( \frac{\omega}{c} \right)_{c \rightarrow 0} = n \times 90^\circ$$



## Exercise 93

Sketch the Bode diagram for the function  $G(s) = \frac{10}{s^2}$



## Bode plot building blocks

**Poles on the real axis**  $G(s) = \frac{1}{\frac{s}{\omega_0} + 1}$ ,  $G(j\omega) = \frac{1}{j\frac{\omega}{\omega_0} + 1}$

The magnitude of  $G(s)$  is

$$|G(j\omega)| = \left| \frac{1}{j\frac{\omega}{\omega_0} + 1} \right| = \left| \frac{1}{j\frac{\omega}{\omega_0} + 1} \times \frac{-j\frac{\omega}{\omega_0} + 1}{-j\frac{\omega}{\omega_0} + 1} \right| \quad (17)$$

$$|G(j\omega)| = \left| \frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 + 1} - j\frac{\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2 + 1} \right| = \sqrt{\left(\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 + 1}\right)^2 + \left(\frac{-\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2 + 1}\right)^2} \quad (18)$$

$$|G(j\omega)| = \sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^2 + 1}{\left[\left(\frac{\omega}{\omega_0}\right)^2 + 1\right]^2}} = \sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2} + 1}}$$

## Bode plot building blocks

**Poles on the real axis**  $G(s) = \frac{1}{\frac{s}{\omega_0} + 1}$ ,  $G(j\omega) = \frac{1}{j\frac{\omega}{\omega_0} + 1}$

$$|G(j\omega)| = \sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2} + 1}}$$

In Decibels, the gain is

$$|G(j\omega)|_{dB} = 20 \log \sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2} + 1}} = 20 \log \left( \frac{\omega^2}{\omega_0^2} + 1 \right)^{-\frac{1}{2}} = \boxed{-20 \log \sqrt{\frac{\omega^2}{\omega_0^2} + 1}}$$

From Equation (18), the phase  $\phi = \tan^{-1} \Im/\Re$  is

$$\phi = \tan^{-1} \left[ -\frac{\frac{\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2 + 1}}{\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 + 1}} \right] = \boxed{\tan^{-1} \left( -\frac{\omega}{\omega_0} \right)}$$

## Bode plot building blocks

$$|G(j\omega)|_{dB} = -20 \log \sqrt{\frac{\omega^2}{\omega_0^2} + 1}$$

$$\phi = \tan^{-1} \left( -\frac{\omega}{\omega_0} \right)$$

**Case 1:**  $\omega \ll \omega_0$ . Thus  $\omega^2/\omega_0^2 + 1 \approx 1$

→ The gain is  $-20 \log(1) = 0$  dB

→ The phase is  $\tan^{-1}(0) = 0^\circ$

**Case 2:**  $\omega = \omega_0$

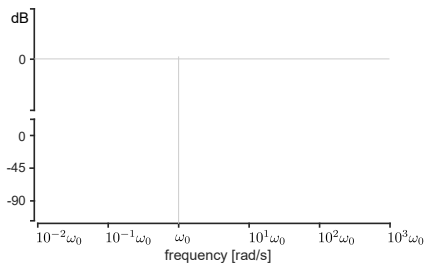
→ The gain is  $-20 \log(\sqrt{2}) = -3$  dB

→ The phase is  $\tan^{-1}(-1) = -45^\circ$

**Case 3:**  $\omega \gg \omega_0$ ,  $\therefore (\omega/\omega_0)^2 \gg 1$

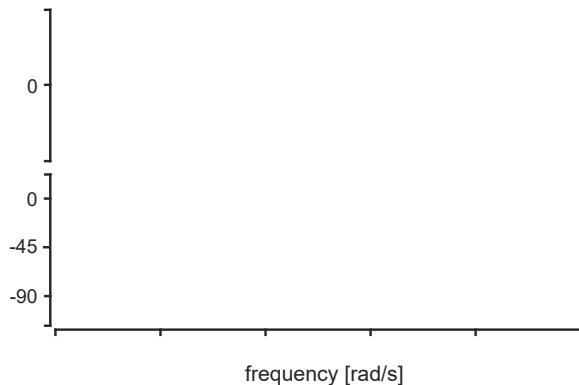
→ The gain is  $-20 \log \left( \sqrt{(\omega/\omega_0)^2} \right)$  dB

→ The phase is  $\tan^{-1}(\infty) = -90^\circ$



## Exercise 94

Sketch the Bode diagram for the function  $G(s) = \frac{1}{s+0.1}$



## Bode plot building blocks

**Zeros on the real axis**  $G(s) = \frac{s}{\omega_0} + 1$ ,  $G(j\omega) = 1 + j\frac{\omega}{\omega_0}$

The gain is

$$|G(j\omega)| = 20 \log \sqrt{\left(\frac{\omega}{\omega_0}\right)^2 + 1} \quad (19)$$

Note that the above is equal the negative gain of a pole on the real axis.

The phase is

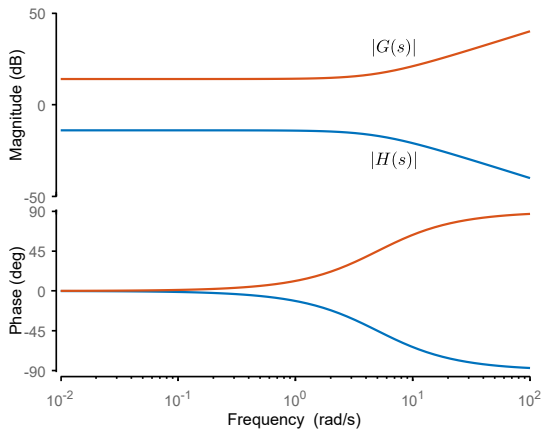
$$\phi = \tan^{-1} \left( \frac{\omega}{\omega_0} \right) \quad (20)$$

Recall that the phase for a real pole was  $\tan^{-1} \left( -\frac{\omega}{\omega_0} \right)$

The real zero is the **negative of a real pole** on the Bode plot

## Bode plot building blocks

$$H(s) = \frac{1}{s+5}, \quad G(s) = s + 5,$$



## Exercise 95

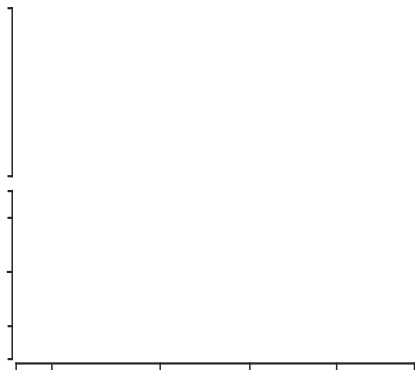
A tendon-operated robotic hand can be implemented using a pneumatic actuator. The actuator can be represented by

$$G(s) = \frac{1000}{(s + 100)(s + 10)} \quad (21)$$

Plot the frequency response of  $G(j\omega)$ . Calculate the magnitude in dB of  $G(j\omega)$  at  $\omega = 10$  rad/s  $\omega = 200$  rad/s.

## Exercise 95 - continued

$$G(s) = \frac{1000}{(s + 100)(s + 10)} \quad (22)$$



## Exercise 95 - continued

$$G(s) = \frac{1000}{(s + 100)(s + 10)} \quad (23)$$

## Exercise 96

Plot the Bode magnitude and phase for the system with the transfer function

$$G(s) = \frac{2000(s + 0.5)}{s(s + 5)(s + 50)}$$

### Procedure

Identify the components of the transfer function for which you know the Bode plot. Draw the Bode plot for each component, then add them up to find the Bode plot of  $G(s)$ .

## Exercise 96 - continued

$$G(s) = \frac{2000(s + 0.5)}{s(s + 5)(s + 50)}$$

→ Constant:

→ Pole at the origin:

→ Real zero:

→ Real pole:

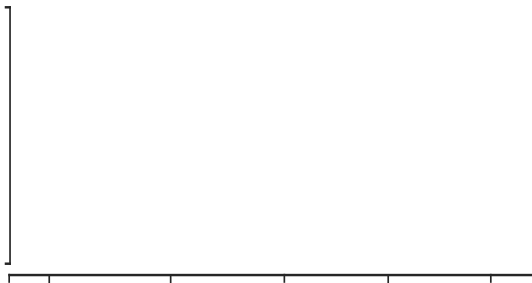
The gain is

$$G = 20 \log \left[ \frac{2000(j\omega + 0.5)}{j\omega(j\omega + 5)(j\omega + 50)} \right]$$

Which can be expressed as

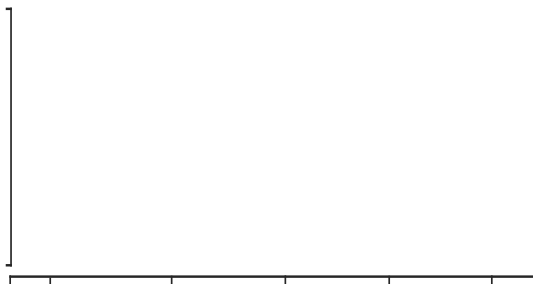
## Exercise 96 - continued

$$G(s) = \frac{2000(s + 0.5)}{s(s + 5)(s + 50)} =$$



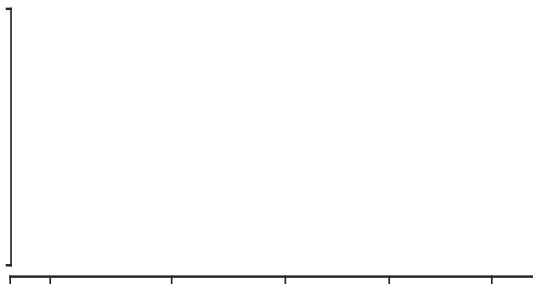
## Exercise 96 - continued

$$G(s) = \frac{4\left(\frac{s}{0.5} + 1\right)}{s\left(\frac{s}{5} + 1\right)\left(\frac{s}{50} + 1\right)}$$



## Exercise 96 - continued

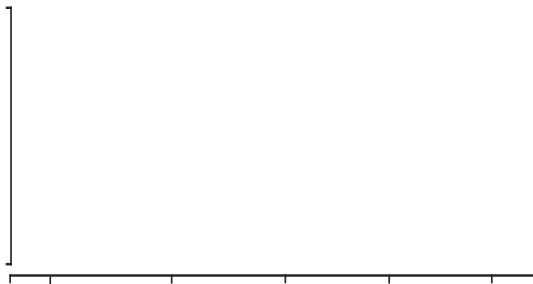
$$G(s) = \frac{4\left(\frac{s}{0.5} + 1\right)}{s\left(\frac{s}{5} + 1\right)\left(\frac{s}{50} + 1\right)}$$



## Exercise 96 - continued

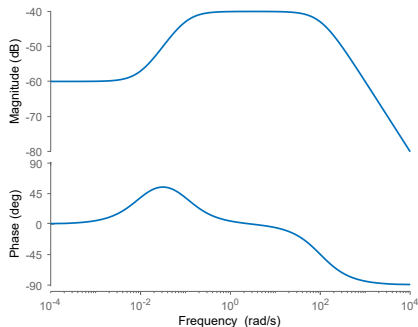
Repeat the magnitude plot for the following function

$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)} =$$

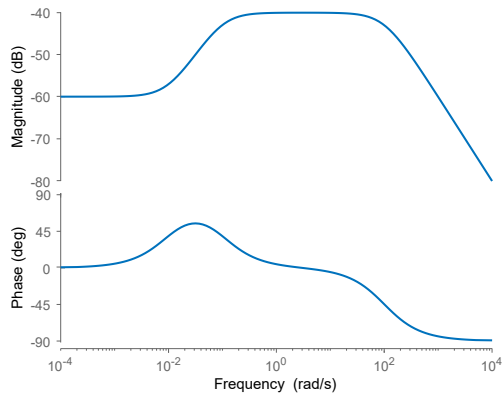


## Exercise 97

The body plot of a system was determined experimentally using a sinusoidal wave excitation. Estimate its transfer function.



## Exercise 97 - continued



## Exercise 98

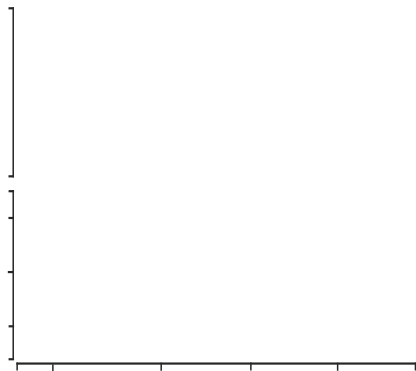
A robotic arm has a joint-control loop transfer function

$$G(s) = \frac{(s + 1)^2}{s(s + 0.1)(s + 10)}.$$

Plot the Bode diagram of the transfer function.

## Exercise 98 - continued

$$G(s) = \frac{(s + 1)^2}{s(s + 0.1)(s + 10)}$$



## Exercise 99

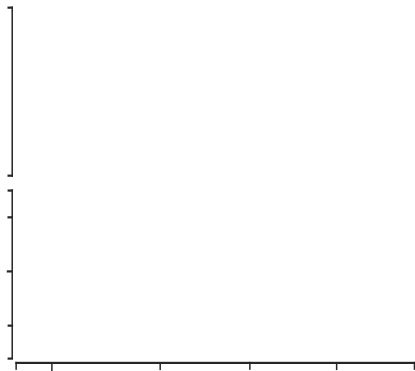
A typical industrial robot has multiple degrees of freedom. A unity feedback position control system for a force-sensing joint has a loop transfer function

$$W(s) = \frac{10s}{\left(\frac{s}{0.1} + 1\right)(s + 1)\left(\frac{s}{80} + 1\right)}.$$

Sketch the Bode diagram of this system.

## Exercise 99 - continued

$$W(s) = \frac{10s}{\left(\frac{s}{0.1} + 1\right)(s + 1)\left(\frac{s}{80} + 1\right)}$$



## Next class...

- More on Bode plots