MECE 3350U
Control Systems

Lecture 9

Dominant Poles and Zeros

Midterm exam - Section 15

When: Monday, Oct 15, 11:10-12:30

What: Lectures 1 to 8

Where: Room split by **first** name:

A-I	J-Z
UL9	UA2120

Prepare your formula sheet (1 page, letter size, both sides)

Everything must be handwritten

Your formula sheet cannot exceed 1 page (letter size), both sides.

Please write your name/student ID on the formula sheet

→ Bring a photo ID or student card.

→ Exam problems are in line with those solved in class, tutorials, and assignments.

→ Office hours during the reading week: As usual.

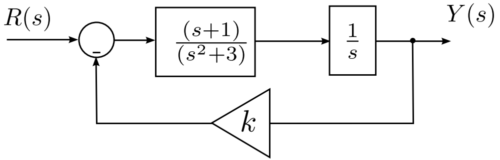
Outline of Lecture 9

By the end of today's lecture you should be able to

- Understand the concept of dominant poles
- Recognize the influence zeros on the transient repose
- Simplify a transfer function to lower orders

Applications

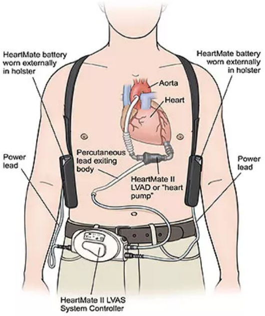
The roll control autopilot of an aircraft has the following structure:



How can we calculate the *k* that yields an overshoot of less than 2%?

Applications

A ventricular assist device is a mechanical pump used to support heart function and blood flow in people with weak or failing hearts.



The model of the heart and pump system results in a third order transfer function. How can we analyse the transient response of the system?

First order system

Consider the response of a first order system to an unit step input:

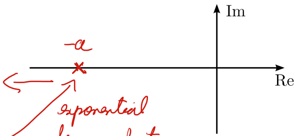
$$X(s) = \frac{1}{s+a}\left(\frac{1}{s}\right)$$

Using partial fraction expansion:

$$X(s) = \frac{1/a}{s} - \frac{1/a}{s+a}$$

The inverse transform yields

$$x(t) = \frac{1}{a}(1 - e^{-at})$$



exponential decays faster as $|a|$ increases.

The transfer function has one pole located at $s = -a$.

$\rightarrow$ How does the magnitude of $s = -a$ influence the transient response?

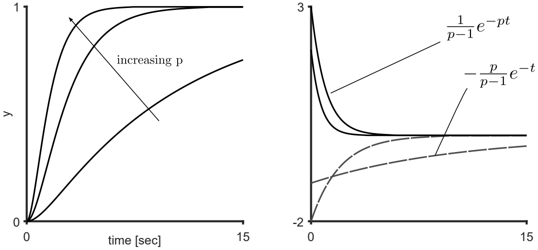
The effect of an additional pole

Let us now examine the step response of

$$X(s) = \frac{p}{(s+1)(s+p)}\left(\frac{1}{s}\right) = \frac{1}{(s+1)(\frac{1}{p}s+1)}\left(\frac{1}{s}\right).$$

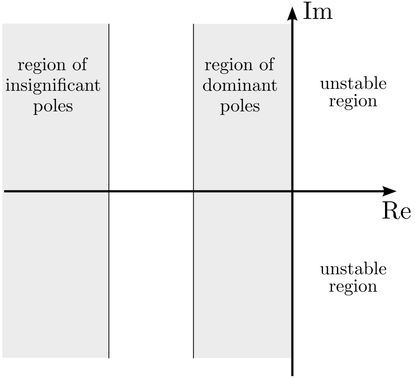
Partial fraction expansion gives:

$$y(t) = 1 - \frac{p}{p-1}e^{-t} + \frac{1}{p-1}e^{-pt}$$



Conclusion: If $p >> 1$, the term $1/(p-1)e^{-pt}$ is negligibly small as $t \to \infty$.

The effect of an additional pole



If the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole, then the pole may be regarded as insignificant.

Second order systems with an additional pole

Consider the 3rd order function

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

Real part of the poles are: $-1/\gamma$ and $-\zeta\omega_n$. Thus, if

$$\left|\frac{1}{\gamma}\right| \geq 10|\zeta\omega_n| \tag{1}$$

The response can be approximated by

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$

Take $\omega_n = 1$, and $\zeta = 0.45$: gives two poles at $s = -0.45 \pm 0.89i$.

Example 1: $\gamma = 1.00 \rightarrow$ Adds a pole to $s = -1$

Example 2: $\gamma = 0.22 \rightarrow$ Adds a pole to $s = -4.5$.

Example 3: $\gamma = 0.10 \rightarrow$ Adds a pole to $s = -10$.

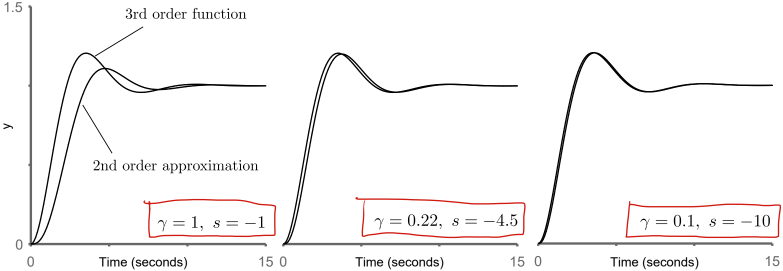
Second order systems with an additional pole

Original 3rd order function:

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

2nd order approximation:

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$



Additional zeros

Consider the transfer function with an additional zero $s = -z$:

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

If $z >> \zeta\omega_n$, the zero will have minimal effect on the step response.

The unit step response of the above equation is:

$\mathcal{L}^{-1} = x(t)$

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\frac{\omega_n^2}{z} s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

If $x(t)$ is the inverse of the first term, than the time response is

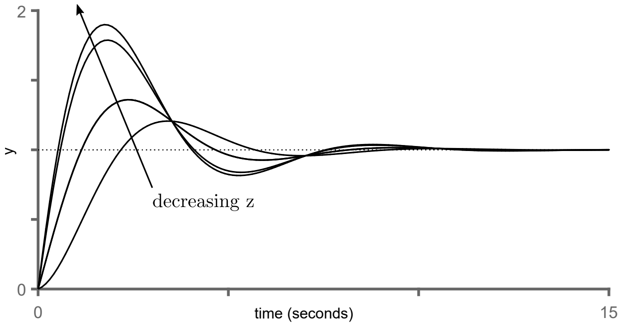
$$\boxed{y(t) = x(t) + \frac{1}{z}\left(\frac{d}{dt}x(t)\right)} \tag{4}$$

Conclusion: The additional zero speeds us transients, making rises and falls sharper.

Additional zeros

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

Consider: $\omega_n = 1,\ \zeta = 0.45,\ z = 0.7,\ 1,\ 10$



Simplification to a lower order

A more precise approach: Match the frequency response.

for information only.

Consider the high order system:

$$G_H(s) = K\frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + 1}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + 1} \tag{6}$$

with $m \geq n$, which is to be mapped to a lower order system

$$G_L(s) = K\frac{c_p s^p + c_{p-1}s^{p-1} + \ldots + c_1 s + 1}{d_g s^g + d_{g-1}s^{g-1} + \ldots + d_1 s + 1} \tag{7}$$

such that $p \leq g \leq n$.

The c and d coefficients of the approximate solution G_L are obtained via

$$M^k = \frac{d^k}{ds^k}M(s) \tag{8}$$

$$\Delta^k = \frac{d^k}{ds^k}\Delta(s) \tag{9}$$

Simplification

for information only.

Let us define

$$M_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} M^k(0) M^{2q-k}(0)}{k!(2q-k)!} \tag{10}$$

$$\Delta_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} \Delta^k(0) \Delta^{2q-k}(0)}{k!(2q-k)!} \tag{11}$$

So that the c and d coefficient are obtained by equating

$$M_{2q} = \Delta_{2q} \tag{12}$$

for $q = 1, 2...$ and up to the number required to solve for the unknowns.

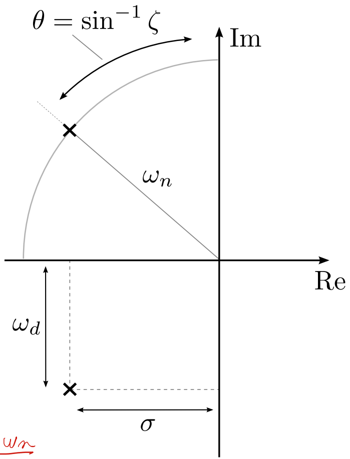
Location of poles

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$



→ Poles are located at a radius ω_n

→ The angle to the imaginary axis is $\theta = \sin^{-1}\zeta$

$|s| = \sqrt{\sigma^2 + \omega_d^2}$

$|s| = \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)}$

$\boxed{|s| = \omega_n}$

$\sin\theta = \frac{\sigma}{\omega_n} = \frac{\zeta\omega_n}{\omega_n}$

$\boxed{\sin\theta = \zeta}$

Exercise 40

A closed-loop control system has a transfer function $T(s)$ as follows

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)(s^2+10s+50)}.$$

Plot the time response to an unit step input when:

→ **(a)** The actual $T(s)$ is used (use Matlab)

→ **(b)** Using the dominant complex poles

→ **(c)** Compare the results

Exercise 40 - continued

(a) The actual function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)\underbrace{(s^2+10s+50)}_{}}.$$

$\Rightarrow s' = -5 \pm 5j$

(b) The approximate transfer function is

$$T(s) = \frac{2500}{(s+50)(s^2+10s+50)}$$

0

$$\boxed{T'(s) = \frac{50}{s^2+10s+50}}$$

!

$50 \gg |-5|$

$\Rightarrow$ can be neglected

$\Rightarrow$ Magnitude of $T'(s)$ and $T(s)$ must be the same.

Exercise 41

A closed-loop control system transfer function as two dominant complete conjugate poles. Sketch the region in the left-hand s-plane where the complex poles should be located to meet the given specifications:

- $\rightarrow$ **(a)** $0.6 \leq \zeta \leq 0.8, \quad \omega_n \leq 10$
- $\rightarrow$ **(b)** $0.5 \leq \zeta \leq 0.707, \quad \omega_n \geq 10$
- $\rightarrow$ **(c)** $\zeta \geq 0.5, \quad 5 \leq \omega_n \leq 10$
- $\rightarrow$ **(d)** $\zeta \leq 0.707, \quad 5 \leq \omega_n \leq 10$
- $\rightarrow$ **(e)** $\zeta \geq 0.6, \quad \omega_n \leq 6$

Exercise 41 - continued

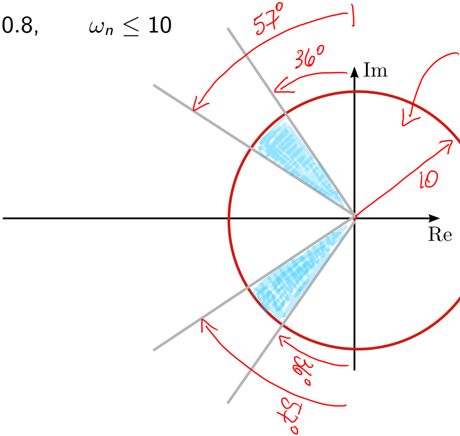
→ **(a)** $0.6 \leq \zeta \leq 0.8, \qquad \omega_n \leq 10$

$\theta_1 = \sin^{-1}(0.6)$

$\theta_1 = 36^\circ$

$\theta_2 = \sin^{-1}(0.8)$

$\theta_2 = 53^\circ$



in the cycle, $\omega_n < 10$ rads

poles must lie in the blue area to meet the specification

Exercise 41 - continued

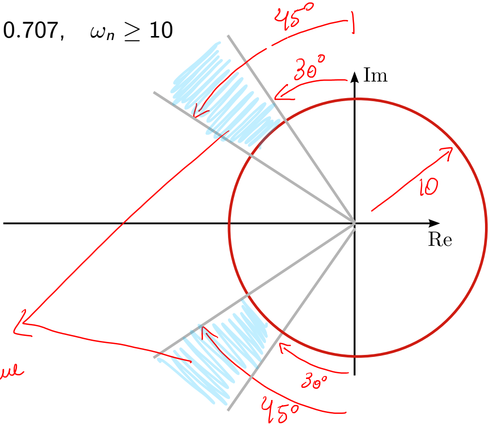
→ **(b)** $0.5 \leq \zeta \leq 0.707, \quad \omega_n \geq 10$

$\theta_1 = \sin^{-1}(0.5)$

$\theta_1 = 30^\circ$

$\theta_2 = \sin^{-1}(0.707)$

$\theta_2 = 45^\circ$



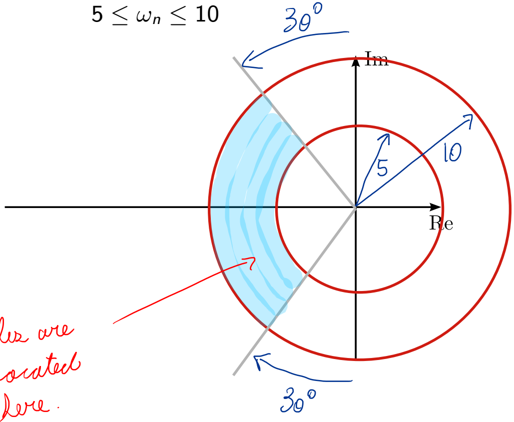
poles are in the blue area.

Exercise 41 - continued

→ **(c)** $\zeta \geq 0.5, \qquad 5 \leq \omega_n \leq 10$

$\theta = \sin^{-1}(0.5)$

$\theta = 30^0$



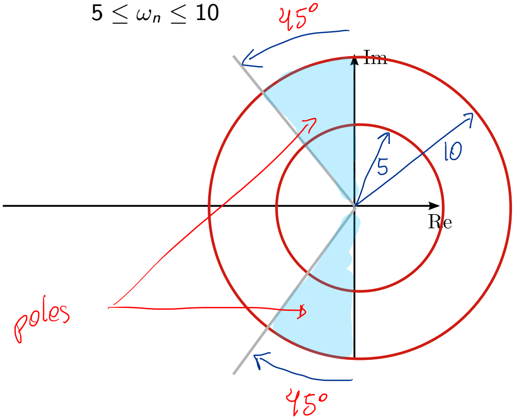
poles are located here.

Exercise 41 - continued

→ **(d)** $\zeta \leq 0.707$, $5 \leq \omega_n \leq 10$

$\theta = \sin^{-1}(0.707)$

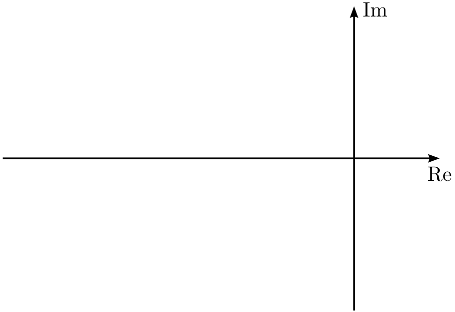
$\theta = 45^\circ$



Exercise 41 - continued

→ **(e)** $\zeta \geq 0.6, \qquad \omega_n \leq 6$

homework



Exercise 42

A closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

→ **(a)** Determine the steady state error for a unit step input.

→ **(b)** Assume that the complex poles dominate and determine the percent overshoot an setting time.

→ **(c)** Plot the actual system response and compare it with **(b)**

→ Matlab code posted on BB

Exercise 42 - continued

(a) Steady-state error for $r(t) = 1$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)} \cdot \frac{1}{s}$$

$$e_{ss} = 1 - \lim_{s \to 0} s\left(\frac{108(s+3)}{(s+9)(s^2+8s+36)} \cdot \frac{1}{s}\right)$$

$$e_{ss} = 1 - \frac{108 \times 3}{9 \times 36}$$

$$\boxed{e_{ss} = 0}$$ (yey)

Exercise 42 - continued

(b) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(\cancel{s}+9)(s^2+8s+36)}.$$

0

$$T'(s) = \frac{12(s+3)}{s^2+8s+36}$$

$\omega_n = \sqrt{36} = 6$ rad/s

$2\zeta\omega_n = 8$

$\rightarrow \zeta = 0.67$

$$P.O. = 100\, e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\boxed{P.O. = 6\%}$$

Exercise 42 - continued

(c) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{6 \times 0.67}$$

$$\boxed{T_s = 1 \text{ sec}}$$

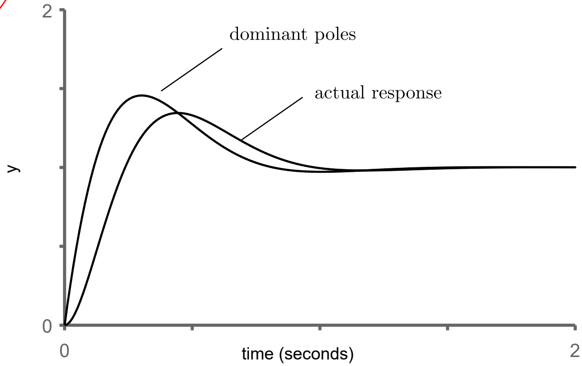
Exercise 42 - continued

T = tf([108 324],[1 17 108 324]);
step(T); stepinfo(T)

H = tf([108/9 324/9],[1 8 36]);
step(H); $\boxed{\text{stepinfo(H)}}$

Matlab posted on Blackboard.

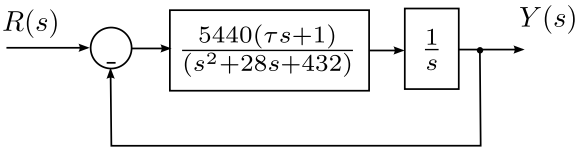
!



Exercise 43

Homework.

Consider the following closed loop system



Where τ can take the values $\tau = 0,\ 0.05,\ 0.1$ or 0.5. For $r(t) = 1$:

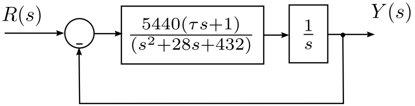
→ **(a)** Record the percent overshoot, rise time, and settling time as τ varies.

→ **(b)** Describe the effects of varying τ.

→ **(c)** Compare the location of the zero with that of the closed-loop poles.

Exercise 43 - continued

The closed loop transfer function



See next slide for solution

Exercise 43 - continued

$$T(s) = \frac{5440(\tau s + 1)}{s^3 + 28s^2 + (432 + 5440\tau)s + 5440}$$

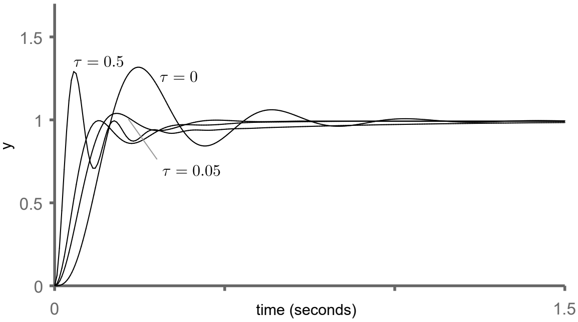
Matlab commands:

```
H = tf([5440*t 5400],[1 28 432+5440*t 5440]);
infostep(H)
damp(H)
```

τ	T_r	T_s	P.O.	zero	pole
0	0.16	0.89	37%	N.A.	-20, $-4 \pm 16j$
0.05	0.14	0.39	4.5%	1/0.05	-10.4, $-8.77 \pm 21j$
0.1	0.10	0.49	0 %	1/0.1	-6.5, $-10.7 \pm 26j$
0.5	0.04	1.05	29.2%	1/0.5	-1.75, $-13.12 \pm 59j$

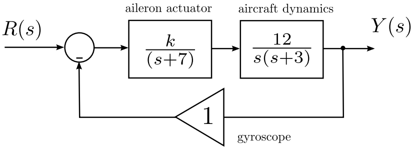
Exercise 43 - continued

```
t = 0;
H1 = tf([5440*t 5400],[1 28 432+5440*t 5440]);
step(H1);
```



Exercise 44

The roll control of an aircraft is shown. The goal is to select a suitable K so that the response to a step command $r(t) = A$ will provide a fast response with an overshoot of less than 20%.



Steps for designing the controller:

$\rightarrow$ **(a)** Determine the closed-loop transfer function

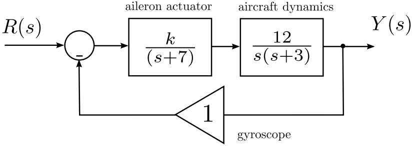
$\rightarrow$ **(b)** Determine the poles for $K = 0.7$, 3, and 6;

$\rightarrow$ **(c)** Using the concept of dominant poles find the expected overshoot

$\rightarrow$ **(d)** Plot the actual response with Matlab and compare it with (c)

Exercise 44 - continued

(a) The closed-loop transfer function



$$Y(s) = \frac{12K}{s(s+3)(s+7)+12K} \cdot R(s)$$

$$Y(s) = \frac{12K}{s^2+10s^2+21s+12K} R(s)$$

Exercise 44 - continued

(b) Finding the poles

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \qquad (13)$$

$s^3+10s^2+21s+12K=0$

$K=0.7 \quad \begin{cases} s_1 = -7.27 \\ s_2 = -2.21 \\ s_3 = -0.52 \end{cases}$

$K=3 \quad \begin{cases} s_1 = -7.91 \\ s_{23} = -1.04 \pm 1.86j \end{cases}$

$K=4 \quad \begin{cases} s_1 = -8.453 \\ s_{23} = -0.74 \pm 2.81j \end{cases}$

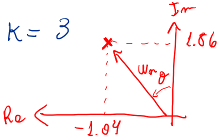
Exercise 44 - continued

(c) Overshoot considering the dominant poles (k =0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{14}$$

$K = 0.3 \rightarrow$ poles are real $P.O = 0$

$K = 3$

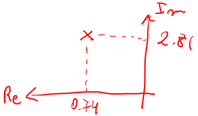


$$\theta = \tan^{-1}\left(\frac{1.04}{1.86}\right) = 29^\circ$$

$$\zeta = \sin(29^\circ) = 0.488$$

$$P.O. = 100\, e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 17\%$$

$K = 6$



$$\theta = \tan^{-1}\left(\frac{0.74}{2.81}\right)$$

$\theta = 14^\circ$, $\zeta = \sin(14^\circ) = 0.25$

$$P.O. = 100\, e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 43\%$$

$$\omega_n = \sqrt{0.74^2 + 2.81^2} = 2.9058 \text{ rad/s}$$

$$T_s = \frac{4}{\omega_n \zeta} = 3.4s$$ //

Exercise 44 - continued

(d) Step-unit response using Matlab

Exercise 44 - continued

(c) Overshoot considering the dominant poles (k =0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{15}$$

See previous slides.

Next class...

- Stability