MECE 3350U
Control Systems

Lecture 8

Transient Response

Midterm exam - Section 21

When: Monday, Oct 15, 9:40-11:00

What: Lectures 1 to 8

Where: Room split by **first** name:

A-J	K-Z
UL9	UA2140

Prepare your formula sheet (1 page, letter size, both sides)

Everything must be handwritten

Your formula sheet cannot exceed 1 page (letter size), both sides.

Please write your name/student ID on the formula sheet

→ Bring a photo ID or student card.

→ Exam problems are in line with those solved in class, tutorials, and assignments.

→ Office hours during the reading week: As usual.

Outline of Lecture 8

By the end of today's lecture you should be able to

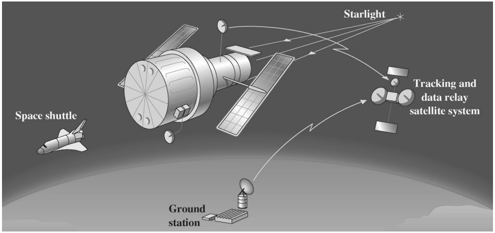
- Understand the concept of transient response
- Recognize the relation between pole location and transient response
- Analyse the transient response of second order systems

Applications

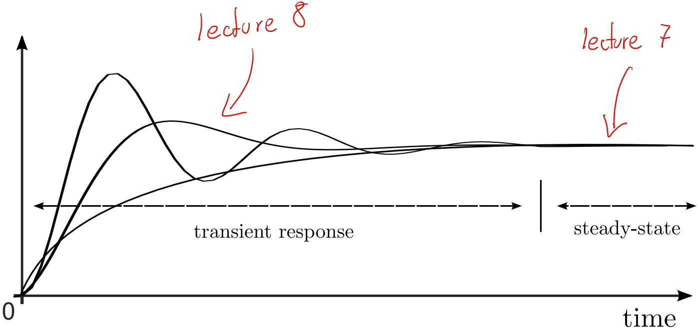
The levitation control system of the train must ensure that the train does not touches the guide. How can we design a controller that reacts as fast as possible with no overshoot?

Applications

The pointing control system of a space telescope is desired to achieve an accuracy of 0.01 minute of arc. How can we limit the steady state error while avoiding transient oscillations?



Transient response

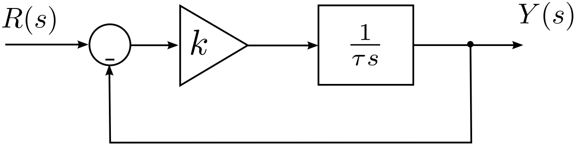


lecture 8

lecture 7

First order systems

Consider the first order closed-loop system shown with a proportional gain *k*



The transfer function $Y(s)/R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{1}{\left(\frac{\tau}{k}\right)s + 1}$$

the higher k, the faster y(t) reaches steady-state

How does *k* influence the transient and steady state response?

To analyse the performance of the system, we need to specify a **standard test input signal**.

Standard test signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

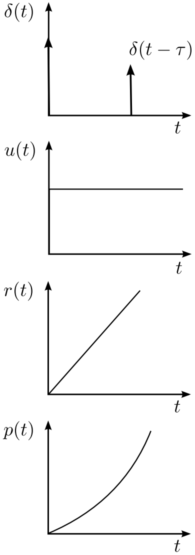
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Temporal response

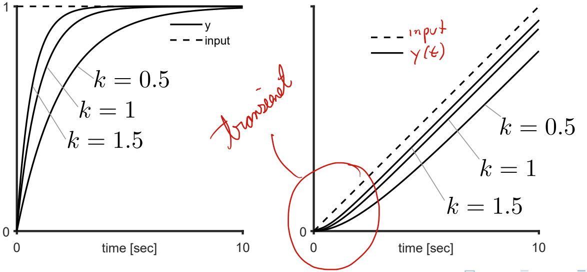
Step response $r(t) = 1$

$$Y(s) = \frac{1}{s}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{k}{\tau}t}$$

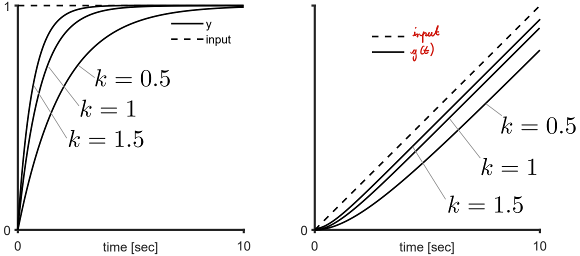
Ramp response $r(t) = t$

$$Y(s) = \frac{1}{s^2}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = t - \frac{\tau}{k}(1 - e^{-\frac{k}{\tau}t})$$

Effects of k for $k > 0$ for $\tau = 1$.



Temporal response - first order system



In a first-order system:

→ *k* reduces the **time constant** of the system

→ The higher *k*, the faster the response

→ What is the maximum control-loop gain *k* ?

Time constant - first order systems

Impulse:

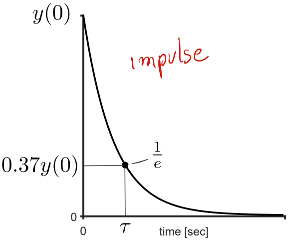
$$H(s) = \frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = y(0)e^{-\frac{t}{\tau}}$$

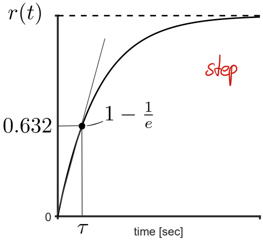
→ When $t = \tau$, the response 37% (1/e) of $y(0)$

Step response

$$H(s) = \frac{1}{s}\frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{t}{\tau}}$$

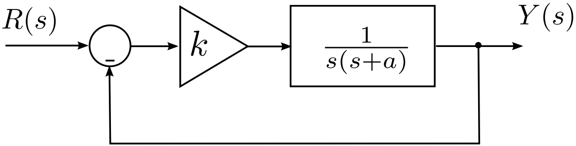
→ When $t = \tau$, the response 67% (1-1/e) of its steady state value





Second-order systems

Consider now the following second order control system:



The transfer function is

$$\frac{Y(s)}{R(s)} = \frac{k}{s^2 + sa + k}$$

We can rewrite the above equation in the standard formulation:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \underset{\times s}{2\zeta\omega_n} + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

Transient response

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

How does k influence the response of the system?

$\rightarrow$ The natural frequency ω_n depends on k

$\rightarrow$ The damping ratio ζ depends on k

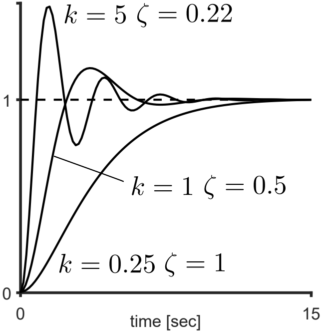
The response for an unit step input when $0 < \zeta < 1$ is

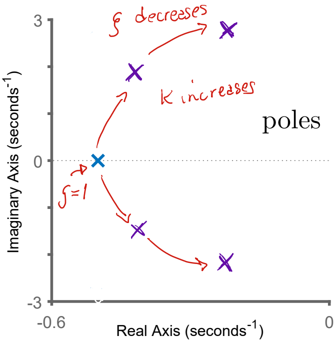
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) t + \cos^{-1}\zeta\right]$$

Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right) \tag{1}$$



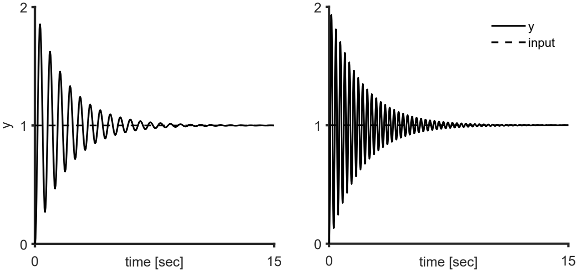


Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

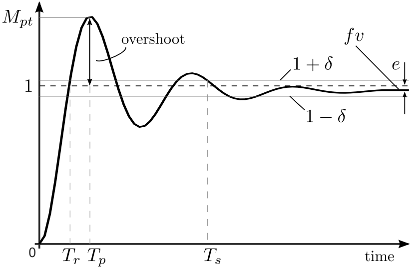
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \cos^{-1}\zeta\right)$$

Natural frequency for $k = 1000$ and $k = 5000$.



How can we evaluate the performance of the controller?

Measures of performance



→ Rise time T_r, peak time T_p, and peak value M_{pt}

→ Settling time T_s: $y(t)$ within 2% of its final value

→ Percent overshoot $P.O.$

→ T_r and T_p characterize the **swiftness** of the response

→ $P.O.$ and T_s characterize the **closeness** of the response to the input

Overshoot

For an unit step input, the percent overshoot is

$$P.O. = \frac{M_{pt} - fv}{fv} \times 100 \qquad (2)$$

→ M_{pt} is the peak value

→ fv is the magnitude of the input

Differentiating Eq (1) and setting it to zero yields the peak time as

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \qquad (3)$$

Replacing (3) into (1) gives the peak response:

$$M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} \qquad (4)$$

Thus, the percentage overshoot ($fv = r(t) = 1$) is

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \qquad (5)$$

Settling time

For an unit step input and $0 < \zeta < 1$, recall that

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1}\zeta\right)$$

When $t = T_s$, the response is within 2% of its final value, thus:

$$e^{-\zeta\omega_n T_s} < 0.02$$

or

$$\zeta\omega_n T_s \approx 4 \tag{6}$$

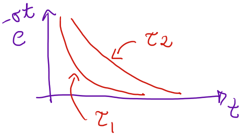
therefore

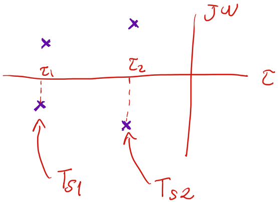
$$T_s = \frac{4}{\zeta\omega_n} = 4\tau \tag{7}$$

where $\tau = 1/\zeta\omega_n$ is the time constant.

The settling time is equal to 4 times the time constant

Settling time





$T_{s2} < T_{s1}$

The characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

has poles:

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$
$$s_1 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

since

$$T_s \approx \frac{4}{\zeta\omega_n} \tag{8}$$

Therefore the settling time is inversely proportional to the real part of the poles.

→ the closer the pole is to the imaginary axis ($\tau \rightarrow 0$), the slower the exponential decay, thus the higher T_s //

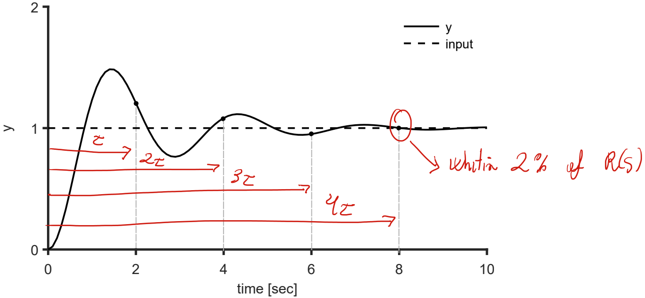
Settling time

Consider the function

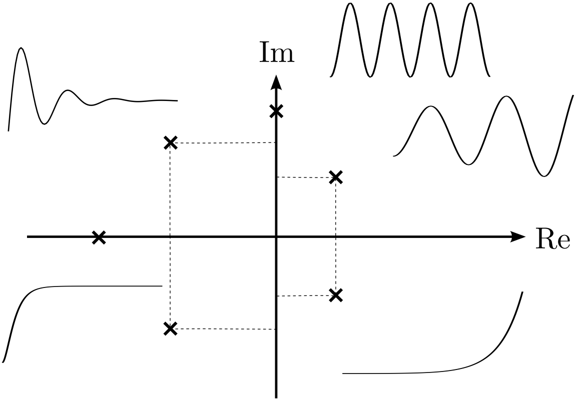
$$H = \frac{5}{s^2 + s + 5}$$

thus: $\omega_n = \sqrt{5}$ and $\zeta = 1/(2\sqrt{5})$. The time constant is

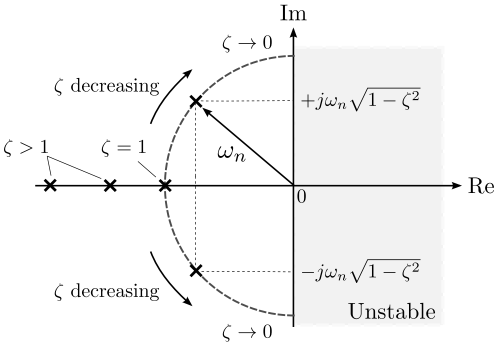
$$\tau = \frac{1}{\zeta \omega_n} = 2 \text{ sec}$$



Poles and transient response



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



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Exercise 35

⚠

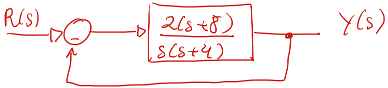
A feedback system with a negative unity feedback has the loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

Determine:

- → **(a)** The closed-loop transfer function
- → **(b)** The time response for an input $r(t) = A$
- → **(c)** The percent overshoot of the response
- → **(d)** The steady state error

Exercise 35 - continued



(a) The closed-loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

$$Y(s) = \frac{2(s+8)}{s(s+4)+2(s+8)} R(s)$$

$$Y(s) = \frac{2(s+8)}{s^2+6s+16} R(s)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{2(s+8)}{s^2+6s+16}}$$

Exercise 35 - continued

$$R(s) = \frac{A}{s}$$

(b) The time response

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

partial fraction

$$Y(s) = \frac{1}{S} - \frac{S+4}{s^2+6s+16}$$

$$\boxed{y(t) = 1 - 1.07e^{-3t}\sin\left[\sqrt{7}t + 1.21\right]}$$

Exercise 35 - continued

(c) The percentage overshoot

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

$\dot{y}(t) = e^{-3t}\left[3.21\sin(\sqrt{7}t + 1.21) - 1.07\sqrt{7}\cos(\sqrt{7}t + 1.21)\right]$

$\dot{y}(t) = 0$

$\boxed{t = 1.003s}$ ← peak time

$y(1.003s) = 1.035 \Rightarrow 3.5\%$ if $A=1$

Exercise 35 - continued

(d) The steady state error

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

$$E(s) = R(s) - Y(s)$$

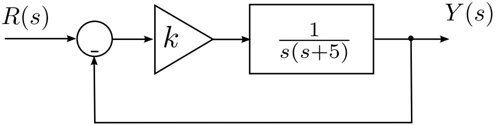
$$E(s) = \frac{A}{s} - \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$\boxed{e_{ss} = 0}$$

Exercise 36

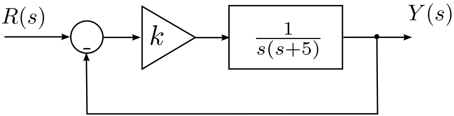
Consider the following block diagram:



→ **(a)** Calculate the steady-state error for a ramp input

→ **(b)** Select *k* that will result in zero overshoot to a step input

Exercise 36 - continued



$R(s) = \frac{A}{s^2}$

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2+5s+K}$$

$$E(s) = R(s) - Y(s)$$

$$E(s) = \frac{A}{s^2} - \frac{K}{s^2+5s+K} \cdot \frac{A}{s^2}$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$e_{ss} = \lim_{s \to 0} s \frac{A}{s^2}\left(1 - \frac{K}{s^2+5s+K}\right) = \lim_{s \to} \frac{A}{\cancel{s}}\left(\frac{\cancel{s^2}+5\cancel{s}+\cancel{K}-\cancel{K}}{s^2+5s+K}\right)$$

$$\boxed{e_{ss} = \frac{5A}{K}}$$

Exercise 36 - continued

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 5s + K}$$

No overshoot

$$\zeta = 1 \; //$$

$$\omega_n = \sqrt{K}$$

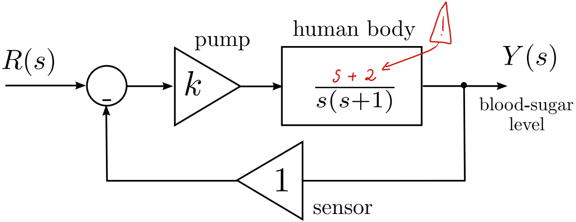
$$2\zeta\omega_n = 5$$

$$\sqrt{K} = \frac{5}{2 \times \underset{\nwarrow \zeta}{1}}$$

$$\boxed{K = 6.25}$$

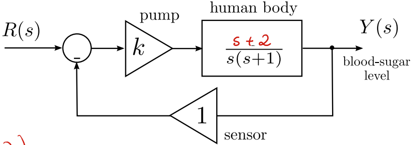
(critically damped)

An insulin pump injection system for diabetic persons has a feedback control as shown.



Calculate a suitable gain k so that the percent overshoot of the step response due to the drug injection is 7%. $R(s)$ is the desired blood sugar level and $Y(s)$ is the actual level. Plot the expected overshoot for different k using Matlab.

Exercise 37 - continued



$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(s+2)}{s(s+1)+K(s+2)}$$

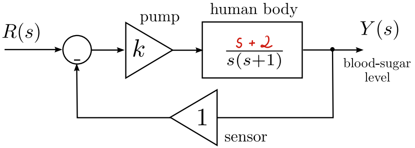
$$\frac{Y(s)}{R(s)} = \frac{K(s+2)}{s^2+\underbrace{s(K+1)}_{2\zeta\omega_n}+\underbrace{2K}_{\omega_n^2}} \quad , \quad \omega_n = \sqrt{2K} \qquad \zeta = \frac{K+1}{2\sqrt{2K}}$$

$$P.O. = 100\, e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}$$

$$0.07 = e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}} \Rightarrow \boxed{\zeta = 0.677}$$

required damping for P.O. = 7%

Exercise 37 - continued



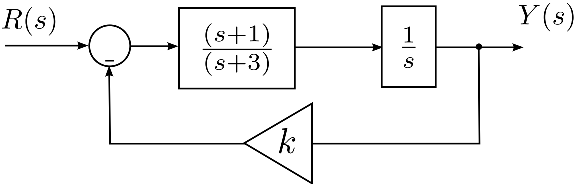
from T(s)

$$2\zeta\omega_n = K+1$$

$$0.677 = \frac{K+1}{2\underbrace{\sqrt{2K}}_{\omega_n}} \longrightarrow \boxed{K \simeq 1}$$

Exercise 38

Consider the following closed loop system



Determine:

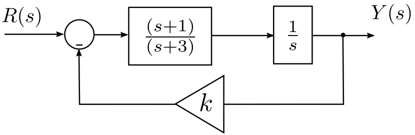
→ **(a)** Determine the closed loop transfer function

→ **(b)** Determine the steady state error to an unit ramp input

→ **(c)** Select k so that the steady state error to a unit step input is zero.

Exercise 38 - continued

(a) Determine the closed loop transfer function

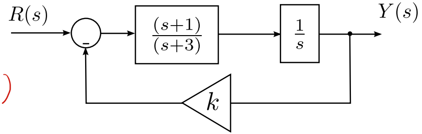


$$\frac{Y(s)}{R(s)} = \frac{\frac{s+1}{s(s+3)}}{1 + K\frac{s+1}{s(s+3)}} = \frac{s+1}{s^2+s(K+3)+K} \quad //$$

Exercise 38 - continued

(b) Determine the steady state error to a unit ramp input

$R(s)=\frac{1}{s}$



$E(s)=R(s)-Y(s)$

$$E(s)=\frac{1}{s^2}\left(1-\frac{s+1}{s^2+s(K+3)+K}\right)$$

$$e_{ss}=\lim_{s\to 0} sE(s)$$

$$e_{ss}=\frac{1}{s}\left(\frac{s^2+s(K+2)+K-1}{s^2+s(K+3)+K}\right)$$

$\longrightarrow$ if $K=1$, $e_{ss}=3$

if $K\neq 1$, $e_{ss}\to\infty$

Exercise 38 - continued

(c) Determine k so that $e = 0$ when $r(t) = 1$

$K - 1 = 0 \rightarrow K = 1$

$$E(s) = -\frac{s^2 + (2+k)s + k - 1}{s^2 + (3+k)s + k}R(s) \tag{10}$$

$\boxed{K=1}$

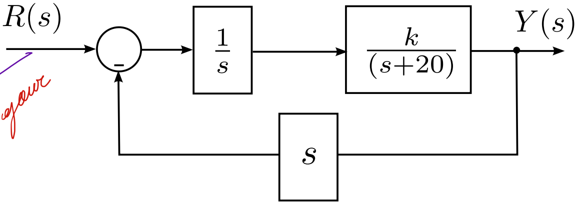
(See previous slide)

$$\lim_{s \to 0} s\,E(s) = 0$$

if $K = 1$ //

Exercise 39

A closed-loop system designed to orient a photovoltaic array towards the direction of maximum solar incident has the following structure:



Homework

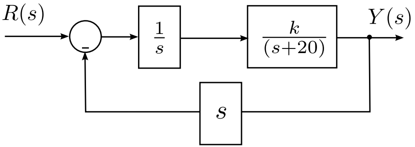
try this on your own...

If $k = 20$, determine:

→ **(a)** The time constant of the closed loop system

→ **(b)** The settling time to within 2% of the final value of the system to an unit step **disturbance**.

Exercise 39 - continued



ⓐ $\tau = \frac{1}{40}$ sec.

ⓑ $p(t) = -\frac{1}{2}\left(1 - e^{-40t}\right)$

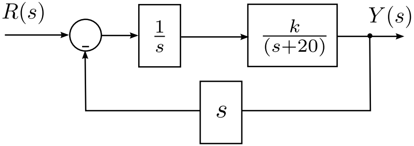
$p(t) = 2\%(1) \times p_{ss}$ ← final value of p_{ss} for $R(s) = \frac{1}{s}$

$$\boxed{p(t) = -0.49}$$

$$-0.49 = -\frac{1}{2}\left(1 - e^{-40t}\right)$$

$$\boxed{t = 0.098s}$$

Exercise 39 - continued



Next class...

- Dominant poles