MECE 3350U
Control Systems

Lecture 5
Effect of Pole Locations

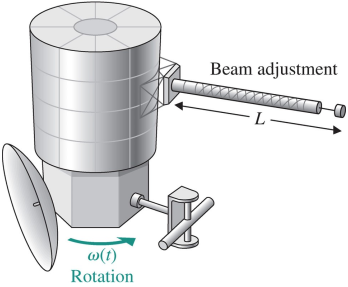
Outline of Lecture 5

By the end of today's lecture you should be able to

- Understand the concept of transient response
- Observe the influence of the pole locations in the temporal response
- Find the temporal response of a system for a given input

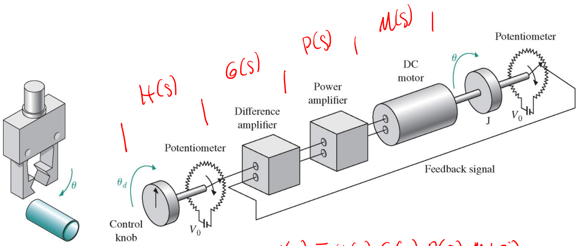
Applications

The rotational velocity of of the satellite is adjusted by changing the length of the beam. How can we determine the shape of the transient response?



Applications

A robot gripper is to be controlled by a DC motor. How can we determine the transient response of the gripper's position?



$$T(s) = H(s)\,G(s)\,P(s)\,M(s)$$

From the last lecture

A transfer function can be written as

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \tag{1}$$

with $n \geq m$. The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\boxed{\lim_{s \to z_i} N(s) = 0} \tag{2}$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

$$\boxed{\lim_{s \to p_i} H(s) = \infty} \tag{3}$$

Poles and zeros

$X(s) = \int x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$

Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2 + 2s + 5}$$

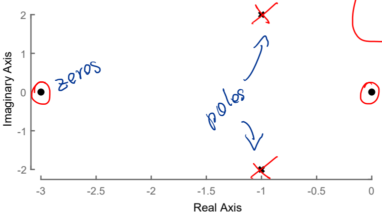
$\underbrace{|X(s)|}$

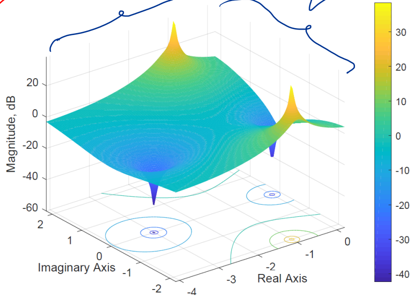
σ

ω

$\rightarrow$ Poles: $-1+2j,\ -1-2j$

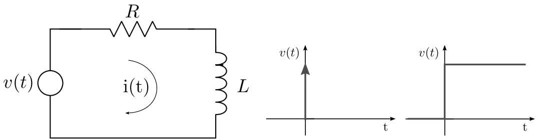
$\rightarrow$ Zeros: $0,\ -3$





First order systems

Consider the RL circuit shown.



$$V(s) = (R + Ls)I(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1/R}{s(L/R) + 1}$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \tag{4}$$

$\rightarrow$ The denominator must be in the form of $\tau s + \mathbf{1}$

First order transfer functions

$H(s) = \frac{1}{s+a} \rightarrow h(t) = e^{-at}$

$H(s) = \frac{1}{s-a} \rightarrow h(t) = e^{+at}$!

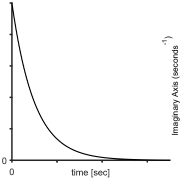
Impulse response: $v(t) = \delta(t) \Rightarrow V(s) = 1$

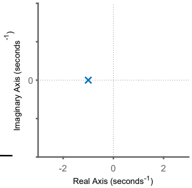
$$I(s) = \frac{1}{R}\frac{1}{\tau s + 1} = \frac{1}{\tau R}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

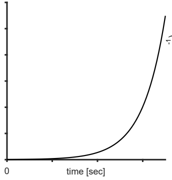
The pole is $s = -1/\tau$. The time response is:

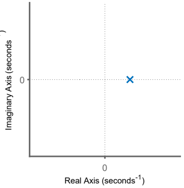
$$i(t) = \frac{1}{\tau R} e^{-\frac{t}{\tau}}$$

converges









unstable.

If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

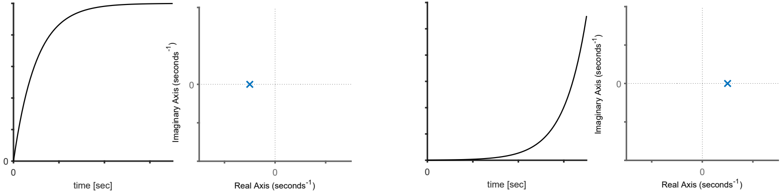
First order transfer functions

Step response: $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$I(s) = \frac{1}{\tau R}\left(\frac{1}{s}\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{k_1}{s}+\frac{k_2}{s+\frac{1}{\tau}}\right)$$

Solving for the partial fraction coefficients: $k_1 = \tau$, $k_2 = -\tau$, thus:

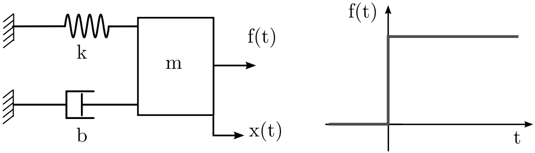
$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$



If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

Second order transfer functions



Transfer function: Standard form

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{m}\left(\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\right)$$

In standard form we have

$$H(s) = \frac{1}{k}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

where:

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

Second order response

Let us now analyse the response to a step input of a second order system

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \quad (6)$$

The poles of the transfer function are:

$$s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad (7)$$

Thus:

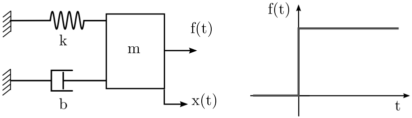
poles $\longrightarrow$

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta - 1}\right)$$

poles can be real or imaginary. depending on ξ

Roots can be real or complex $\Rightarrow$ Inverse transformation ?

Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{8}$$

Case 1: $\zeta \geq 1$, (lots of damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta - 1}\right)$$

if $\zeta > 1$, $s_1, s_2 \in \mathbb{R}$

thus $X(s) = \frac{1}{k}\frac{1}{s}\frac{1}{(s+a)(s+b)}$

$a, b, \in \mathbb{R}$.

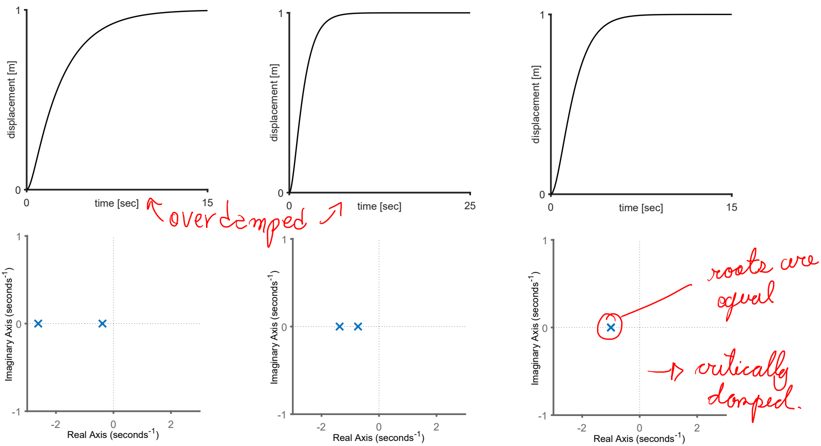
Roots are negative real numbers. Partial fraction expansion yields:

$$X(s) = \frac{1}{m}\left(\frac{k_1}{s} + \frac{k_2}{(s + a_1)} + \frac{k_3}{(s + a_2)}\right) \tag{9}$$

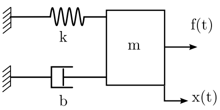
Overdamped system

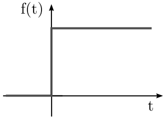
Example: $m = 1$ kg, $k = 1$ N/m

$b = 3$ Ns/m, $\zeta = 1.5$; $b = 2.1$ Ns/m, $\zeta = 1.05$; $b = 2$ Ns/m, $\zeta = 1$.



Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{10}$$

Case 2: $0 < \zeta < 1$, (some damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta - 1}\right)$$

Roots are complex conjugate numbers with a negative real part. Thus:

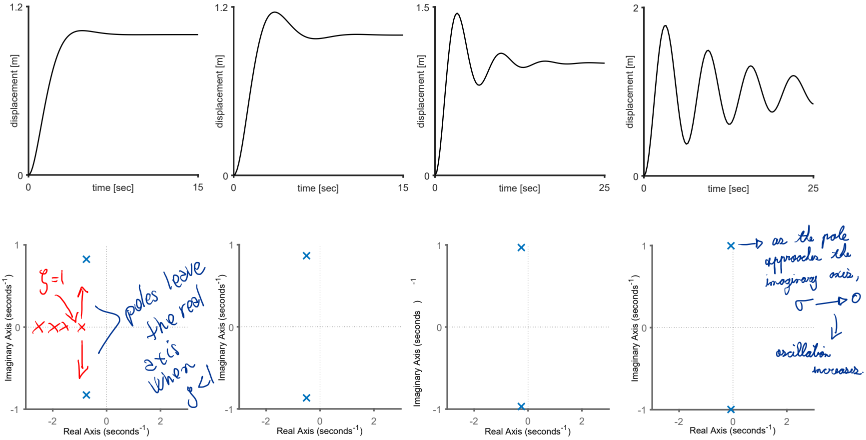
converges if $\zeta > 0$

$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\boxed{e^{-\zeta\omega_n t}}\sin\left(\omega_n\sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \tag{11}$$

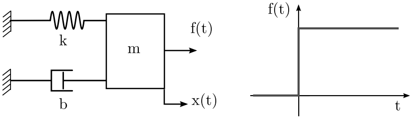
!

Underdamped system

$b = 1.5,\ \zeta = 0.75$ $\quad b = 1,\ \zeta = 0.5;$ $\quad b = 0.5,\ \zeta = 0.25,$ $\quad b = 0.1,\ \zeta = 0.05$



Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + \underbrace{2\zeta\omega_n s}_{=0} + \omega_n^2}\right) \tag{12}$$

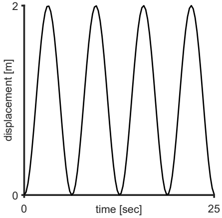
Case 3: $\zeta = 0$, (no damping)

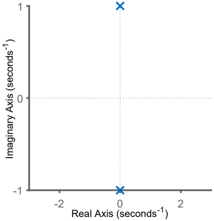
$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta - 1}\right)$$

Roots are purely complex conjugate numbers. Thus:

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) \rightarrow x(t) = \frac{1}{k}[1 - \cos(\omega_n t)] \tag{13}$$

Undamped system





if $\zeta = 0$, poles are purely imaginary and lie on the imaginary axis $(\sigma = 0)$ (no exponential)

The frequency of oscillation for of an undamped system is called the natural frequency.

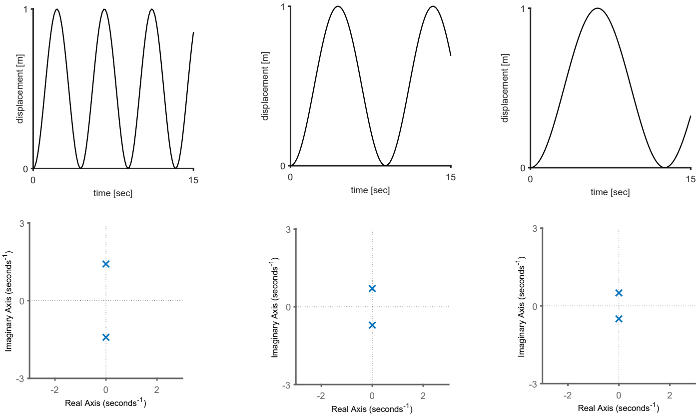
In our example:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{14}$$

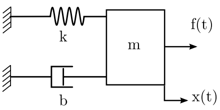
Natural frequency

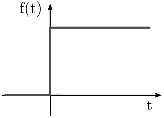
The frequency the system oscillates when $\zeta = 0$. Example: $b = \zeta = 0$, $k = 1$.

$m = 1$, $\omega_n = 1$ rad/s; $m = 2$, $\omega_n = 0.71$ rad/s; $m = 4$, $\omega_n = 0.5$ rad/s;



Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{15}$$

Case 4: $\zeta < 0$, (hypothetical negative damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta - 1}\right), \quad s_2 = \omega_n\left(-\zeta - \sqrt{\zeta - 1}\right)$$

Roots (real or imaginary) have positive real parts. Possible solutions are:

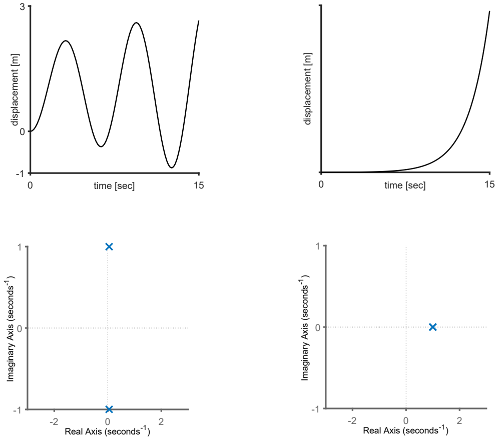
$$x(t) = k(1 + k_2 e^{s_1 t} + k_3 e^{s_2 t}) \tag{16}$$

increases over time

$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{|\zeta|\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \tag{17}$$

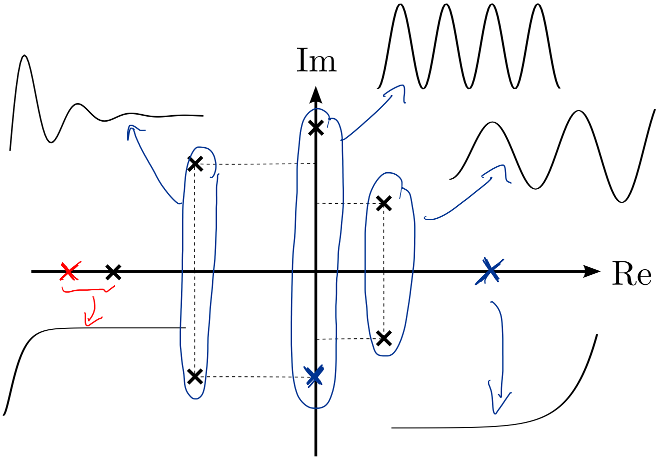
Unstable system

$b = -0.1$ Ns/m; $b < -1$ Ns/m;

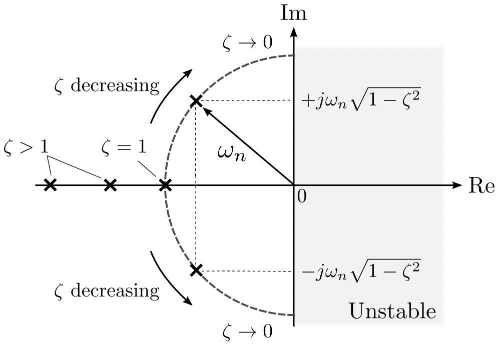


unstable : real part of the poles is positive (at least one)

Summary



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



Location of poles in the s-plane

For a second order system, the poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

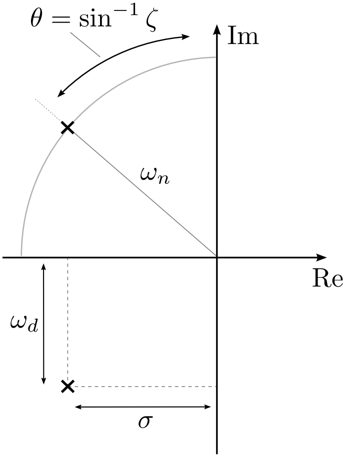
where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\rightarrow$ The magnitude of s is

$$|s| = \sqrt{(\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta}\right)^2} = \omega_n$$

$\rightarrow$ The angle to the imaginary axis is

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} \rightarrow \theta = \sin^{-1}\zeta$$



Exercise 21

Discuss the correlation between the poles of

$$H(s) = \frac{2s + 1}{s^2 + 2s + 5} \tag{18}$$

and the impulse response of the system and find the exact impulse response.

Procedure:

$\rightarrow$ Calculate the damping ratio and the natural frequency

$\rightarrow$ Calculate inverse transform

Homework

Exercise 21 - continued

$$H(s) = \frac{2s+1}{s^2+2s+5}$$

poles are

$s^2 + 2s + 5 = 0$

$$\boxed{\begin{aligned} s &= -1+2i \\ s' &= -1-2i \end{aligned}}$$

$\omega_n = \sqrt{5}$

$2\zeta\omega_n = 2$

$\boxed{\zeta = 0.44}$ ⚠

$$H(s) = \frac{2s+1}{(s+1-2i)(s+1+2i)} = \frac{A}{s+1+2i} + \frac{B}{s+1-2i}$$

⚠

$A(s+1-2i) + B(s+1+2i) = 2s+1$

$$\left\{\begin{aligned} A + B &= 2 \\ A - 2Ai + B + 2Bi &= 1 \end{aligned}\right. \quad \text{solving} \rightarrow \quad A = 1 - \frac{1}{4}i \qquad B = 1 + \frac{1}{4}i$$

Exercise 21 - continued

$$H(s) = \frac{1-\frac{1}{4}i}{s+1+2i} + \frac{1+\frac{1}{4}i}{s+1-2i}$$

$$\mathcal{L}^{-1}$$

$$h(t) = \left(1-\frac{1}{4}i\right)e^{(-1-2i)t} + \left(1+\frac{1}{4}i\right)e^{(-1+2i)t}$$

$$h(t) = \left(1-\frac{1}{4}i\right)e^{-t}e^{-2it} + \left(1+\frac{1}{4}i\right)e^{-t}e^{2it}$$

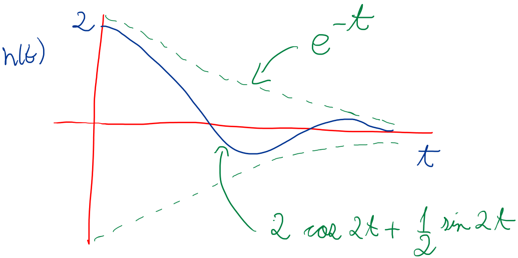
$$h(t) = e^{-t}\left[e^{-2it} + e^{2it} + \frac{1}{4}i\left(e^{-2it} + e^{2it}\right)\right]$$

Exercise 21 - continued

$$\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \quad , \quad \sin\alpha = \frac{1}{2i}\left(e^{i\alpha} + e^{-i\alpha}\right)$$

thus,

$$h(t) = e^{-t}\left(2\cos(2t) + \frac{1}{2}\sin(2t)\right) \quad //$$



Exercise 22

Consider the system of the form

$$H(s) = \frac{9}{s^2 + bs + 9} \tag{19}$$

Calculate and sketch the response to $\boxed{\text{a~~n~~ unit step input}}$ for the following cases

$\rightarrow b = 9$

$\rightarrow b = 0$

$\rightarrow b = 2$

$\rightarrow b = 6$

$v(t) = 1$

$V(s) = \frac{1}{s}$

Exercise 22 - continued

$\omega_n = \sqrt{9} = 3$ rad/s

$\zeta = \frac{9}{2 \cdot 3} = 1.5$

$\rightarrow$ real distinct roots $\rightarrow$ overdamped.

$$H(s) = \underbrace{\frac{9}{s^2 + 9s + 9}}_{s^2 + 2\zeta\omega_n + \omega_n^2} \frac{1}{s} \tag{20}$$

roots are

$s^2 + 9s + 9 = 0$

$\begin{cases} s = 7.854 \\ s = 1.146 \end{cases}$

$$H(s) = \frac{A}{s + 7.854} + \frac{B}{s + 1.146} + \frac{C}{s}$$

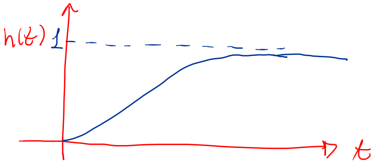
solving partial fraction

$A = 0.171$

$B = -1.171$

$C = 1$

$$\boxed{h(t) = 0.171 e^{-7.854t} - 1.171 e^{-1.146t} + 1}$$



Exercise 22 - continued

$\boxed{b=0}$

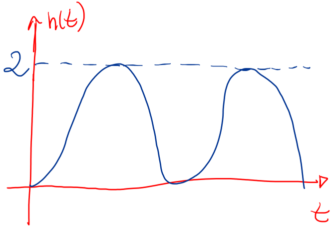
$$H(s) = \frac{9}{s^2+9}\frac{1}{s} \tag{21}$$

$$H(s) = \frac{As+B}{s^2+9} + \frac{C}{s}$$

$A = -1$
$B = 0$
$C = 1$

$$H(s) = \frac{1}{s} - \frac{s}{s^2+9} \quad \mathcal{L}^{-1}$$

$$\boxed{h(t) = 1 - \cos(3t)}$$



Exercise 22 - continued

$\boxed{b=2}$

$$H(s) = \frac{9}{s^2+2s+9}\frac{1}{s} \tag{22}$$

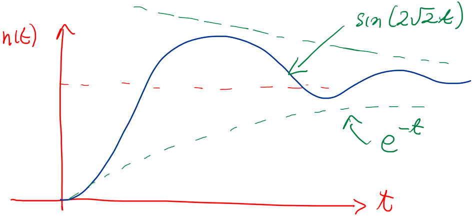
$\omega_n = 3\,rad/s$

$2\zeta\omega_n = 2$

$\zeta = \frac{1}{3} < 1 \rightarrow$ poles are complex conjugate

$$\boxed{h(t) = \frac{9}{2\sqrt{2}} e^{-t} \sin(2\sqrt{2}\, t)}$$

see Exercise 21.
Development is the same.



Exercise 22 - continued

critically damped

$\omega_n = 3$

$2\zeta\omega_n = 6$ $\boxed{\zeta = 1}$

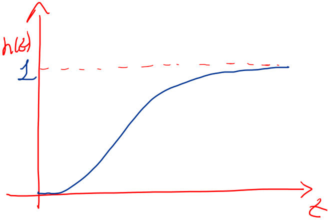
$$H(s) = \frac{9}{s^2 + 6s + 9}\frac{1}{s} \tag{23}$$

roots are $-3, -3$.

$$H(s) = \frac{9}{(s+3)^2} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{(s+3)^2} + \frac{C}{s+3}$$

$A = 1$
$B = -3$
$C = -1$

$$\boxed{h(t) = 1 - 3te^{-3t} - e^{-3t}}$$



Next class...

- Block diagrams