MECE 3350U
Control Systems

Lecture 4

Transfer Functions

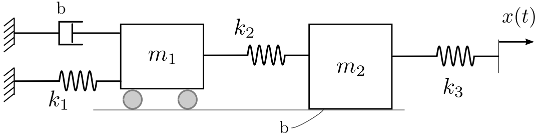
Outline of Lecture 4

By the end of today's lecture you should be able to

- Understand the concept of transfer functions
- Find the transfer function of a given system
- Find the temporal response of a system for a given input

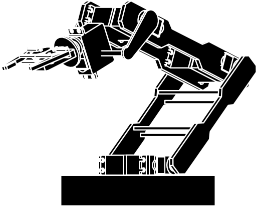
Applications

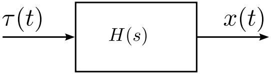
If the spring is stretched to a point $x(t) = 5$ mm, held, then released at time $t = 0$, how does the position of m_1 evolve in time?



Applications

What torque must be applied to each of the robot joints so that end-effector moves along a given trajectory with a given speed?

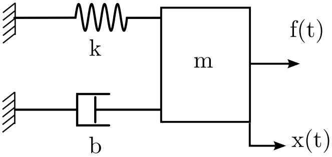




Input/output relation

Transfer function: A relation between the input and output of a linear system

Example: Input: force $f(t)$, output: displacement $x(t)$



$$m\frac{d^2x}{dt} + b\frac{dx}{dt} + kx = f(t)$$

Taking the Laplace transform:

$$ms^2X(s) + bsX(s) + KX(s) = F(s)$$

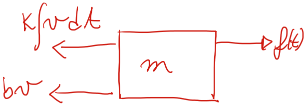
The transfer function is

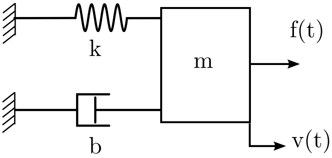
$$X(s)[ms^2 + bs + K] = F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + K} \quad // \tag{1}$$

Input/output relation

Example: Input: force $f(t)$, output: $\boxed{\textbf{velocity}}$ $v(t)$





The dynamic model is $m\frac{dv}{dt} = f(t) - K\int v\,dt - bv$ (2)

Laplace transform of (2) is

$$msV(s) + bV(s) + K\frac{1}{s}V(s) = F(s)$$

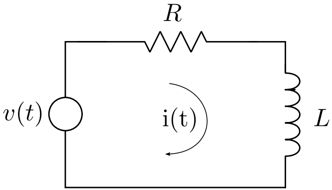
The transfer function is

$$H(s) = \frac{V(s)}{F(s)} = \frac{s}{ms^2 + bs + K} \quad (3)$$

derivative of Eq (1)!

Input/output relation

Input: Voltage $v(t)$, output: Current $i(t)$



$$v(t) = Ri(t) + L\frac{di(t)}{dt} \tag{4}$$

Taking the Laplace transform of (4):

$$V(s) = R\,I(s) + L\,s\,I(s)$$

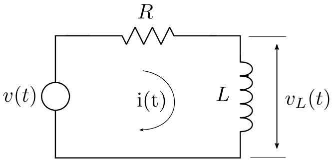
$$V(s) = I(s)\,[R + Ls]$$

The transfer function is

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R}$$

Input/output relation

Input: Voltage $v(t)$, **output**: $v_L(t)$



$$v(t) = Ri(t) + L\frac{di(t)}{dt}, \qquad v_L = L\frac{di(t)}{dt} \tag{5}$$

Taking the Laplace transform of (5):

$V(s) = [R + Ls]\, I(s)$

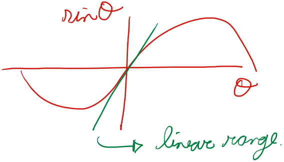
$V_L(s) = Ls\, I(s)$

$I(s) = \frac{V_L(s)}{Ls}$

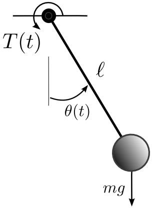
The transfer function is

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R}, \qquad \frac{V_L(s)}{V(s)} = \frac{1 \; Ls}{Ls + R} \rightarrow I(s)$$

Input/output relation



The equation of motion of the simple pendulum is



$$m\frac{d^2\theta(t)}{dt^2} + m\frac{g}{\ell}\underbrace{\sin\theta}_{\theta} = T(t) \tag{6}$$

For small angles, $\sin\theta \approx \theta$, in the frequency domain:

$$ms^2\Theta(s) + \frac{mg}{\ell}\Theta(s) = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{ms^2 + \frac{mg}{\ell}} \tag{7}$$

Transfer function poles and zeros

Transfer function: A rational function in the complex variable $s = \sigma + j\omega$:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \tag{8}$$

The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\lim_{s \to z_i} N(s) = 0 \tag{9}$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

$$\lim_{s \to p_i} H(s) = \infty \tag{10}$$

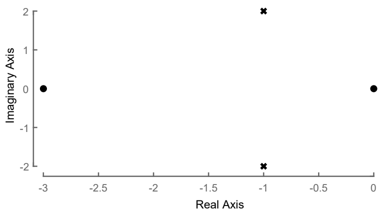
Poles and zeros

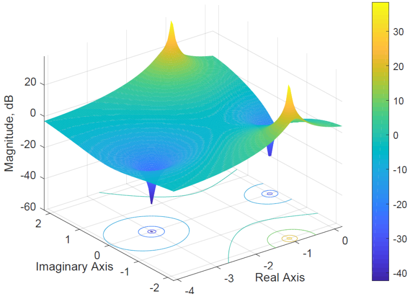
Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2+2s+5}$$

$\rightarrow$ Poles: $-1+2j,\ -1-2j$

$\rightarrow$ Zeros: $0,\ -3$





First order transfer functions

Order: The number of the highest derivative in the denominator (power of s)

Standard form of a first order system:

$$G(s) = k\frac{1}{\tau s + 1}$$

Characteristic equation: The denominator of the transfer function

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1}{R}\,\frac{1}{\frac{L}{R} + 1}$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \tag{11}$$

$\rightarrow$ The denominator must be in the form of $\tau s + \mathbf{1}$

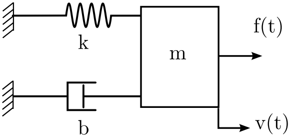
Second order transfer functions

Standard form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

Where: ω_n is the natural frequency, ζ is the damping ratio.

We will come back to these definitions in the next lecture.



$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{m}\frac{1}{\left(s^2 + \frac{b}{m}s + \frac{K}{m}\right)} \Rightarrow \frac{1}{m}\cdot\frac{m}{K}\,\frac{K/m}{s^2 + \frac{b}{m}s + \frac{K}{m}}$$

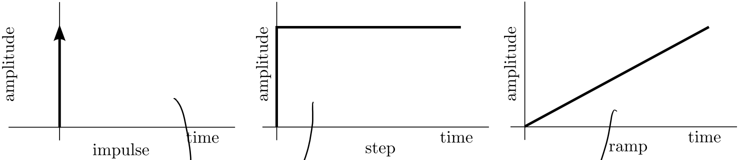
$$= \frac{1}{K}\,\frac{K/m}{s^2 + b/m\, s + K/m}$$

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

Temporal response

Step 1: Define the input signal in Laplace domain

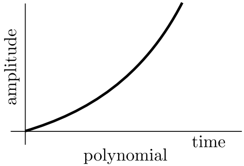


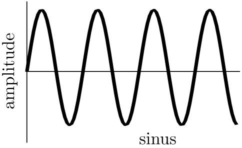
input signal	time domain	frequency domain
impulse	$r(t) = \delta(t)$	$R(s) = 1$
step	$r(t) = a$	$R(s) = a\frac{1}{s}$
ramp	$r(t) = at$	$R(s) = a\frac{1}{s^2}$

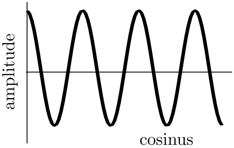
a is a constant

Temporal response

Step 1: Define the input signal in Laplace domain







input signal	time domain	frequency domain
polynomial	$r(t) = at^n$	$R(s) = a\frac{n!}{s^{n+1}}$
sine	$r(t) = \sin(at)$	$R(s) = \frac{a}{s^2+a^2}$
cosine	$r(t) = \cos(at)$	$R(t) = \frac{s}{s^2+a^2}$

a is a constant

there is a typo in the notes.

Temporal response

Step 2: Replace the input signal in the transfer function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{13}$$

For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k} \times 1 \tag{14}$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right) \tag{15}$$

For a sinusoidal input $f(t) = 5\sin(t)$ N:

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{5}{s^2 + 1}\right) \tag{16}$$

1, not 25. (typo)

Temporal response

Step 3: Calculate the inverse transform of the resulting function

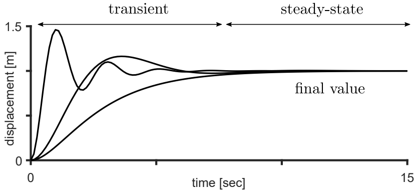
For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\right\} \quad (17)$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right)\right\} \quad (18)$$

and so on.



Steady-state value

Final value theorem: Gives the steady-state value without computing the inverse transform.

If the function converges, i.e., the poles of $sX(s)$ have negative real parts, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{19}$$

For a step type input, the mass spring damper system settles at

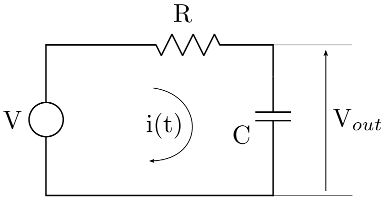
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \cancel{s} \left\{ \frac{1}{ms^2 + bs + k} \left(\frac{1}{\cancel{s}} \right) \right\} = \frac{1}{K} \tag{20}$$

For a impulse input, the RL system converges at

$$\lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s) = \lim_{s\to 0} s \left\{ \frac{1}{Ls + R} \right\} = 0 \tag{21}$$

Exercise 16

Find the transfer function $H(s)$ between the input voltage V and the output voltage V_{out}.



Procedure:

$\rightarrow$ Find the differential equation for the current

$\rightarrow$ Find the equation for the output voltage

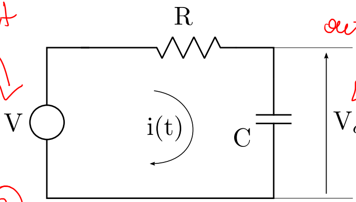
$\rightarrow$ Calculate the transfer function

Exercise 16 - continued

$$v(t) = i(t)R + \frac{1}{C}\int i(t)\,dt$$

$$V(s) = \left[R + \frac{1}{Cs}\right] I(s) \qquad (1)$$

input



output

$$v_{out}(t) = \frac{1}{C}\int i\,dt \rightarrow V_{out}(s) = \frac{1}{Cs} I(s) \qquad (2)$$

$$(2) \rightarrow I(s) = V_{out}(s)\,C.s. \rightarrow (1)$$

$$V(s) = \left[R + \frac{1}{Cs}\right] V_{out}(s)\,C.s$$

$$V(s) = (RCs + 1)\,V_{out}(s)$$

$$\frac{V_{out}(s)}{V(s)} = \frac{1}{RCs+1}$$

final value.

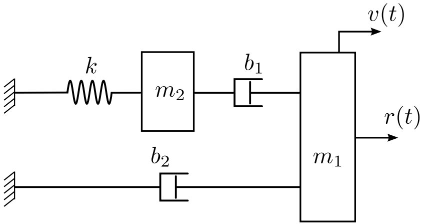
for an impulse $\rightarrow \begin{cases} v(t) = \delta(t) \\ V(s) = 1 \end{cases}$

$$\lim_{t\to\infty} v_{out}(t) = V_{out}(s) \times s \Big|_{s\to 0}$$

$$\lim_{t\to\infty} v_{out}(t) = \left(\frac{1}{RCs+1} \times 1\right) \times s \Big|_{s\to 0} = 0$$

Exercise 17

Find the transfer function $H(s) = \frac{V(s)}{R(s)}$ between the force $r(t)$ and the velocity of mass m_1.



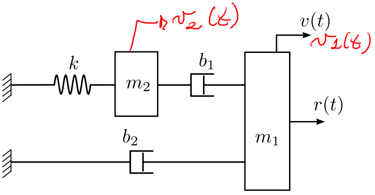
Procedure:

$\rightarrow$ Find the differential equation the velocity of each mass

$\rightarrow$ Calculate the Laplace transform

$\rightarrow$ Calculate the transfer function

Exercise 17 - continued



Mass 1

$$r(t) - b_2 v_1 - b_1(v_1 - v_2) = m\dot{v}_1$$

$$R(s) - b_2 V_1(s) - b_1(V_1(s) - V_2(s)) = msV_1(s)$$

$$\boxed{[m_1 s + b_1 + b_2]V_1(s) - b_1 V_2(s) = R(s) \qquad (1)}$$

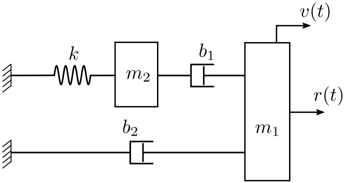
Mass 2

$$b_1(v_1 - v_2) - b_1\int v_2\,dt = m_2\frac{dv_2}{dt}$$

$$m_2 s V_2(s) + b_1(V_2(s) - V_1(s)) + \frac{K}{s}V_2(s) = 0$$

$$\boxed{-b_1 V_1(s) + \left(m_2 s + b_1 + \frac{K}{s}\right)V_2(s) = 0 \qquad (2)}$$

Exercise 17 - continued



$$① \quad [m_1 s + (b_1 + b_2)]V_1(s) - b_1 V_2(s) = R(s)$$

$$② \quad -b_1 V_1(s) + \left(m_2 s + b_1 + \frac{k}{s}\right) V_2(s) = 0$$

$$\rightarrow V_2(s) = \frac{b_1}{m_2 s + b_1 + \frac{K}{s}} V_1(s) \quad ③$$

③ in ① gives

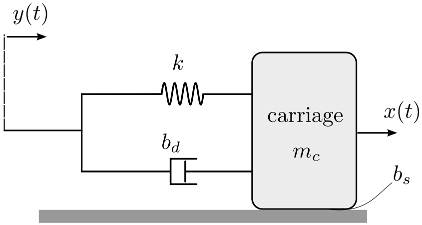
$$[m_1 s + (b_1 + b_2)] V_1(s) - b_1 \left(\frac{b_1}{m_2 s + b_1 + K/s}\right) V_1(s) = R(s)$$

solving.

$$\boxed{\frac{V_1(s)}{R(s)} = \frac{m_2 s^2 + b_1 s + K}{(m_1 s + b_1 + b_2)(m_2 s^2 + b_1 s + K) - b_1^2 s}}$$

Exercise 18

A high precision positioning slide is shown in the figure. The drive shaft friction is $b_d = 0.65$, the drive shaft spring constant is $k = 1.8$, $m_c = 1$, and the slide friction is $b_s = 0.9$.



Determine:

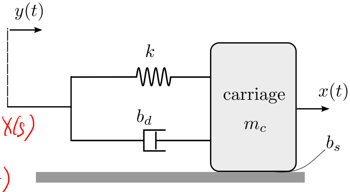
→ Find the transfer function $H(s) = X(s)/Y(s)$.

→ Calculate the natural frequency, damping ratio, the poles, and zeros of $H(s)$

→ Find the steady-state value for a step input

→ Plot the step response of using Matlab

Exercise 18 - continued

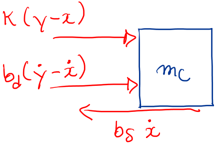


$$m\ddot{x} = k(y - x) + b_d(\dot{y} - \dot{x}) - b_s\dot{x}$$

$$m s^2 X(s) = K\,Y(s) - K\,X(s) + b_d s Y(s) - b_d s X(s) - b_s s X(s)$$

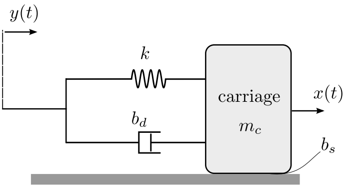
$$m s^2 X(s) + (b_s + b_d) s X(s) + K X(s) = b_d s Y(s) + K Y(s)$$

$$X(s)\left[ms^2 + (b_s + b_d)s + K\right] = Y(s)\left[b_d s + K\right]$$



$$\boxed{\frac{Y(s)}{X(s)} = \frac{b_d s + K}{ms^2 + (b_s + b_d)s + K}}$$

Exercise 18 - continued



$$\frac{Y(s)}{X(s)} = \frac{b_d s + K}{m s^2 + (b_s + b_d)s + K}$$

for a step input $X(s) = \frac{1}{s}$

$$Y(s) = \frac{b_d s + K}{m s^2 + (b_s + b_d)s + K} \times \frac{1}{s}$$

final value $\Rightarrow$ $s\,Y(s)\Big|_{s \to 0}$

$$\underset{t \to \infty}{x(t)} = s\,Y(s)\Big|_{s \to 0} = \cancel{s} \cdot \frac{b_d s + K}{m s^2 + (b_s + b_d)s + K} \cdot \frac{1}{\cancel{s}}\Bigg|_{s \to 0} = \frac{K}{K} = 1$$

Exercise 18 - continued - Using Matlab

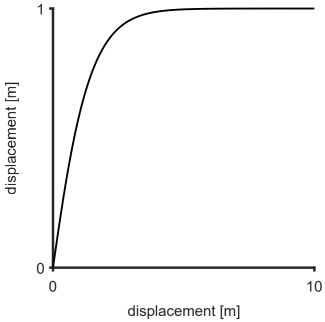
replace with numbers given in the problem.

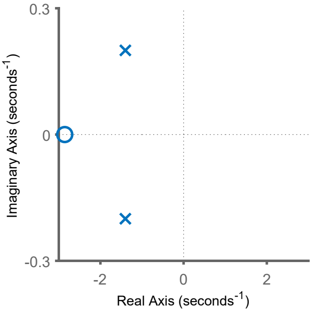
H = tf([0.7 2],[1 2.8 2]) → Transfer function

damp(H) → Natural frequency and damping

step(H,10) → Step response

pzplot(H) → Location of zeros and poles





Exercise 19

Calculate the natural frequency and damping ratio of the following transfer function

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

Determine:

$\rightarrow$ Write the transfer function in standard form

$\rightarrow$ Find the steady-state value for a step input

$\rightarrow$ Calculate the natural frequency and damping ratio

Exercise 19 - continued

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

$$T(s) = \frac{0.525 \times 10^7}{s^2 + 0.8\,10^3 + 0.525 \times 10^7}$$

$$w_n^2 = 0.525 \times 10^7, \quad \boxed{w_n = 2291 \text{ rad/s}}$$

$$2\xi w_n = 0.8 . 10^3$$

$$\xi = \frac{0.8 . 10^3}{2 \times 2291} \rightarrow \boxed{\xi = 0.17}$$

for an step input

$$s\,T(s)\Big|_{s \to 0}$$

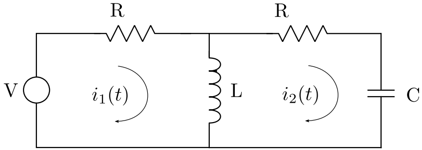
$$s \frac{1.05 \times 10^7}{2s^2 + 1.6 . 10^3 s + 1.05 \times 10^7} \cdot \frac{1}{s}\Bigg|_{s \to 0}$$

$$= 1$$

Exercise 20

Find the transfer function $G(s) = \frac{I_2(s)}{V(s)}$ of the circuit shown. Then, calculate the step response of the circuit using Matlab. Take $R = 10\ \Omega$, $C = 0.001$ F, $L = 0.1$ H, $V = 5$ V.

Homework
Try this on your own.



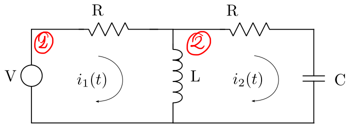
Determine:

→ Find the transfer function $H(s) = I_2(s)/V(s)$.

→ Find the steady-state value for a step input

→ Plot the step response of using Matlab

Exercise 20 - continued



$$V(s) = I_1(s)R + Ls\left(I_1(s) - I_2(s)\right) \quad (1)$$

$$I_1(s) = \frac{V + I_2 L}{R + sL}$$

replace

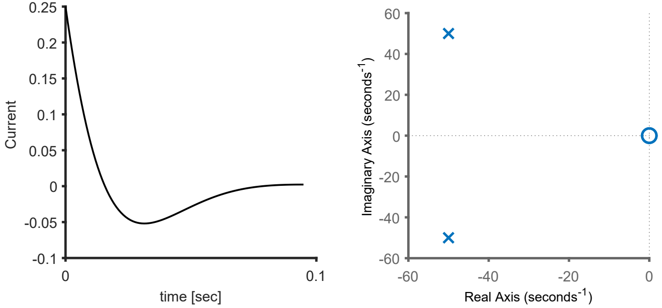
$$\left[I_2(s) - I_1(s)\right]sL + I_2(s)R + \frac{1}{Cs} I_2(s) = 0$$

$$I_2(s)\left[sL + R + \frac{1}{Cs}\right] - sL\left(\frac{V + I_2 + L}{R + sL}\right) = 0$$

$$R_2 + \frac{1}{Cs} + Ls - \frac{L^2 s^2}{R + sL} = \frac{Ls}{R + sL} V(s) \cdot \frac{1}{I_2}$$

$$\boxed{\frac{I_2(s)}{V(s)} = \frac{LCs^2}{s^2(2RCL) + s(L + R^2C) + R}}$$

Exercise 20 - continued - Using Matlab



Next class...

- Effect of pole locations