MECE 3350U
Control Systems

Lecture 3

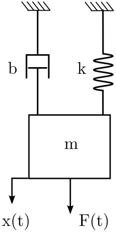
Laplace Transform

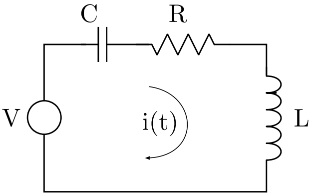
Outline of Lecture 3

In today's lecture we will

- Review the principles of the Laplace transformation
- Apply the Laplace transformation to a system

Applications



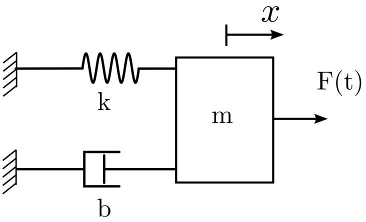


$$F(t) = m\ddot{x} + b\dot{x} + kx$$

$$V(t) = \frac{1}{C}\int i\,dt + iR + L\frac{di}{dt}$$

Input/output relation

Transfer function: A relation between the input and output of a given linear system



$$F \longrightarrow \boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx} \longrightarrow x$$

system

How can we evaluate the temporal response to the system?

Laplace transformation

The time-response solution can be obtained using the Laplace transform.

differential equation | | solution

$$\boxed{\text{time domain}} \xrightarrow{\mathcal{L}} \boxed{\text{frequency domain}} \xrightarrow{\mathcal{L}^{-1}} \boxed{\text{time domain}}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \qquad F(s) = X(s)[ms^2 + bs + k] \qquad x(t) = Ke^{-\alpha t}\sin(\beta t + \theta)$$

$\rightarrow$ Obtain the linearised differential equations

$\rightarrow$ Obtain the Laplace transformation of the differential equation

$\rightarrow$ Solve for the variable of interest

Laplace transformation

A mass-spring system is governed by the differential equation

$$m\frac{d^2x(t)}{dt^2} + Kx(t) = F(t) \tag{1}$$

Solutions of (1) can either be

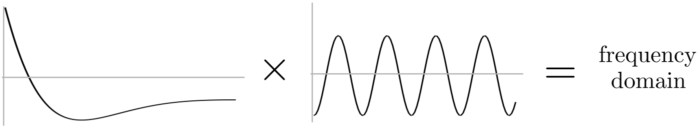
$\rightarrow$ Exponential: $x(t) = e^{at}$ with $a \in \mathcal{R}$

$\rightarrow$ Sinusoidal: $x(t) =\ \sin(at) = e^{-j\omega t}$

or both

Fourier transform: Analyses the frequency response of a system

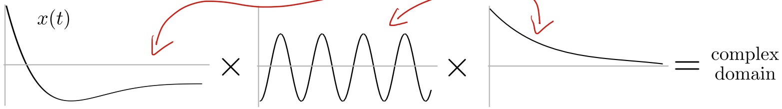
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega}tdt \tag{2}$$



Where is the exponential term??

Laplace transformation

$$X(\sigma,\omega)=\int_{-\infty}^{\infty}\left[x(t)\cdot e^{-\sigma t}\cdot e^{-j\omega t}\right]dt$$



The standard form of the Laplace transform is:

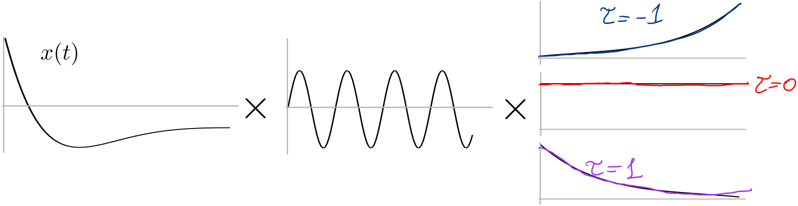
$$X(\sigma,\omega)=\int_{-\infty}^{\infty}\left[x(t)\cdot e^{\overbrace{(-\sigma-j\omega)}^{-s}t}\right]dt \tag{3}$$

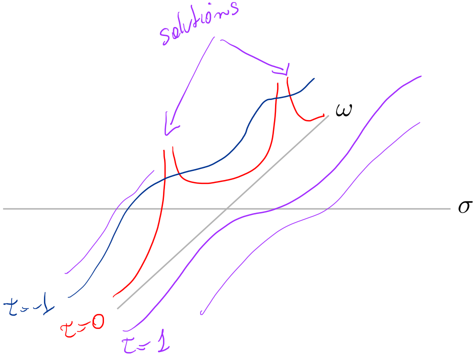
where the complex variable $s=\sigma+j\omega$ has:

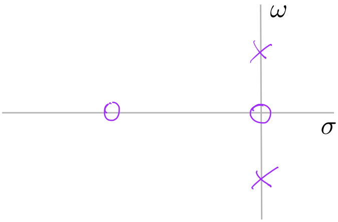
→ A real portion σ, which corresponds to the exponential response

→ An imaginary portion ω, which corresponds to the sinusoidal response

Laplace transformation

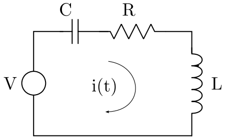


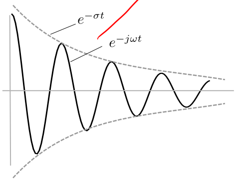


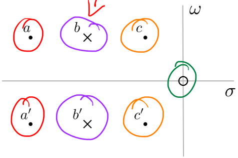


Laplace transformation

assuming that the solution is σ, ω, at b



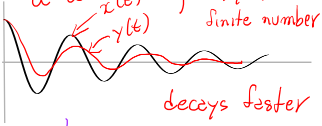




a - a'

$x(t)$ $y(t)$

$\int x(t)y(t)$ finite number

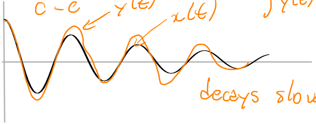


decays faster

c - c'

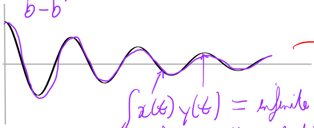
$y(t)$ $x(t)$

$\int y(t)x(t) =$ finite



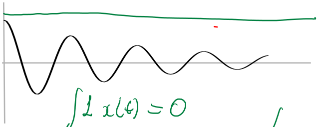
decays slower

b - b'



$\int x(t)y(t) =$ infinite

→ this is the solution



$\int 1\, x(t) = 0$

→ zero!

$$\frac{s}{(s+b')(s+b)}$$

Laplace transformation

The Laplace transformation for a function of time $f(t)$ is

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \mathscr{L}\{f(t)\} \tag{4}$$

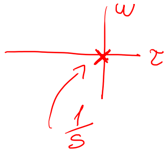
where $s = \sigma + j\omega$.

The inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j}\int_{\tau - j\infty}^{\tau + j\infty} F(s)e^{st}ds \tag{5}$$

Transfer function: The ratio of the Laplace transform of the output variable of to the input variable.

Laplace transformation



What is the Laplace transform of $f(t) = 1$?

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \mathscr{L}\{f(t)\}$$

$$F(s) = \int_0^{\infty} 1\, e^{-st}\, dt = \lim_{A \to \infty} \int_0^{A} e^{-st} dt \to \lim_{A \to \infty} \left[-\frac{1}{s} e^{-st} \right]_0^A$$

$$\lim_{A \to 0} \left[-\frac{1}{s} e^{-sA} + \frac{1}{s} \right] \longrightarrow \frac{1}{s} \,//$$

$$\left(\lim_{A \to 0} e^{-sA} = 0 \right)$$

Laplace transformation

$f(t)$	$F(s)$
Impulse function $\delta(t)$	1
Step function $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace transformation

$\mathcal{L}\{\dddot{x}(t)\} = s^3 X(s) - s^2 x(0) - s\, x(0) - \ddot{x}(0)$

$\mathcal{L}\{\ddot{x}(t)\} = s^2 X(s) - s\, x(0) - \dot{x}(0)$

$\mathcal{L}\{\dot{x}(t)\} = x X(s) - x(0)$

$f(t)$	$F(s)$
$\frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2}\dot{f}(0) - \ldots - f^{k-1}(0)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{1}{s}\int f(t)dt\Big\|_{t=0}$

The Laplace variable s can be considered to be the differential operator:

$$s \Rightarrow \frac{d}{dt} \tag{6}$$

And the integral operator:

$$\frac{1}{s} \Rightarrow \int_0^t dt \tag{7}$$

Laplace transform properties

Linearity

$$\mathscr{L}\{\alpha x(t)\} = \alpha \mathscr{L}\{x(t)\} = \alpha X(s) \tag{8}$$

$$\mathscr{L}\{\alpha x(t) + \beta y(t)\} = \alpha X(s) + \beta Y(s) \tag{9}$$

Time shift

$$x(t-\tau) = X(s)e^{-s\tau} \tag{10}$$

Initial value theorem

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) \tag{11}$$

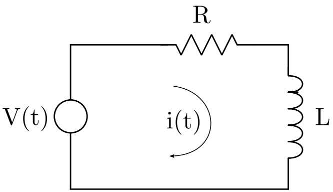
Final value theorem

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)} \tag{12}$$

Laplace transform

Consider all initial conditions to be zero.

$V(s) = (R + Ls) I(s)$



$$\mathscr{L}\{v(t)\} = \mathscr{L}\left\{Ri(t) + L\frac{di(t)}{dt}\right\} \tag{13}$$

$$V(s) = RI(s) + Ls\, I(s) \tag{14}$$

$V(s)$ is called the forcing function: a term that is only a function of time.

Common forcing signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

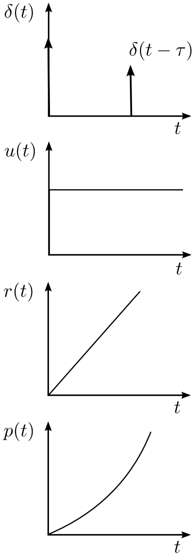
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Partial fraction decomposition

Consider the function $F(s)$ given by

$$F(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{p=0}^{p} a_p s^p} \tag{15}$$

with A and B being polynomials **and p** $>$ **k**.

How to split up a complicated fraction into known forms such as:

$$F(s) = \frac{c_1}{s + a_1} + \frac{c_2}{s + a_2} \dots \frac{a_p}{s + c_p} \ ?$$

Such that:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{c_1}{s + a_1} + \frac{c_2}{s + a_2} \dots + \frac{c_p}{s + a_p}\right\}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{c_1}{s + a_1}\right\} + \mathscr{L}^{-1}\left\{\frac{c_2}{s + a_2}\right\} \dots + \mathscr{L}^{-1}\left\{\frac{c_p}{s + a_p}\right\}$$

Partial fraction decomposition

For each factor in the denominator, the term in the decomposition is:

Factor in denominator	Term in partial decomposition
$as + b$	$\frac{c}{as+b}$
$(as + b)^k$	$\frac{c_1}{as+b} + \frac{c_2}{(as+b)^2} + \dots \frac{c_k}{(as+b)^k}$
$as^2 + bx + d$	$\frac{c_1 s+c_2}{as^2+sb+d}$
$(as^2 + bx + d)^k$	$\frac{c_1 s+e_1}{as^2+sb+d} + \frac{c_2 s+e_2}{(as^2+sb+d)^2} + \dots \frac{c_k s+e_k}{(as^2+sb+d)^k}$

Partial fraction decomposition

$$\frac{3s+11}{(s+3)(s+2)} \qquad \frac{a}{s+3} + \frac{b}{s+2} \rightarrow \frac{a(s+2)+b(s+3)}{\cancel{(s+2)}(s+3)} = \frac{3s+11}{(s+3)\cancel{(s+2)}}$$

$$as + 2a + bs + 3b = 3s + 11$$

$$\left\{\begin{array}{l} \cancel{s}(a+b) = 3\cancel{s} \\ 2a + 3b = 11 \end{array}\right\} \text{solving:} \qquad \left\{\begin{array}{l} b = 5 \\ a = -2 \end{array}\right.$$

thus $\quad F(s) = \dfrac{-2}{s+3} + \dfrac{5}{s+2}$

$$f(t) = -2e^{-3t} + 5e^{-2t}$$

Partial fraction decomposition

$$\frac{s^2+15}{(s+3)^2(s^2+3)} \rightarrow \underbrace{\frac{a}{(s+3)} + \frac{b}{(s+3)^2} + \frac{cs+d}{s^2+3}}_{\text{comom} \rightarrow (s+3)^2(s^2+3)}$$

$$a(s+3)(s^2+3) + b(x^2+3) + (cs+d)(s+3)^2 = s^2+15$$

solving $\quad a=\frac{1}{2}, \quad b=2, \quad c=-\frac{1}{2}, \quad d=\frac{1}{2}$

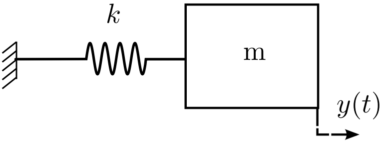
thus: $\quad \frac{1}{2(s+3)} + \frac{2}{(x+3)^2} + \frac{-x+1}{2(s^2+3)}$

Laplace transformation

$F(s)$	$f(t)$
1	Unit impulse $\delta(t)$
$\frac{1}{s}$	Unit step function $u(t)$
$\frac{1}{s^2}$	Unit ramp t
$\frac{n!}{s^{n+1}}$	t^n with $n \in \mathbb{N}^+$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{s}{(s+a)(s+b)}$	$\frac{1}{b-a}\left(be^{-bt} - ae^{-at}\right)$

Exercise 11

If the mass is released from rest when the spring is stretched by $y(0) = \alpha$, calculate its position $y(t)$ as a function of time.



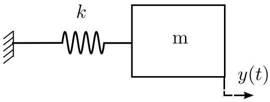
Procedure:

$\rightarrow$ Find the differential equation

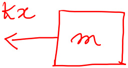
$\rightarrow$ Calculate the Laplace transform

$\rightarrow$ Determine the temporal response using the inverse transformation

Exercise 11 - continued



Given $y(0) = \alpha$, $\dot{y}(0) = 0$. Determine $y(t)$.



$$-ky = m\ddot{y}$$

$$m\ddot{y} + ky = 0 \rightarrow m\left[s^2 Y(s) - s\alpha\right] + kY(s) = 0$$

$$ms^2 Y(s) - ms\alpha + kY(s) = 0$$

$$Y(s)\left[ms^2 + k\right] = ms\alpha$$

$$Y(s) = \frac{ms\alpha}{ms^2 + k} \rightarrow \alpha \frac{s}{s^2 + \frac{k}{m}} \rightarrow \mathcal{L}^{-1} = \alpha \cos\left(\sqrt{\frac{k}{\omega}}\, t\right)$$

from the Laplace table:

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

$$\omega^2 = \frac{k}{m} \rightarrow \omega = \sqrt{\frac{k}{\omega}}$$

Exercise 12

Consider the following differential equation in the frequency domain

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

Determine:

$\rightarrow$ The final value of the function $f(t)$ when $t \rightarrow \infty$

$\rightarrow$ The function $f(t)$

Exercise 12 - continued

Final value

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left(\frac{\cancel{s}\,10}{\cancel{s}(s+1)(s+10)}\right) = \lim_{s\to 0}\left(\frac{10}{(s+1)(s+10)}\right) = \frac{10}{10} = 1$$

Exercise 12 - continued

Final value

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

See previous slide.

Exercise 12 - continued

Partial fraction expansion

$$F(s) = \frac{10}{s(s+1)(s+10)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+10}$$

$$a(s+1)(s+10) + b(s)(s+10) + c(s)(s+1) = 10$$

$$\begin{cases} s^2(a+b+c) = 0 \\ s(11a+10b+c) = 0 \\ 10a = 10 \end{cases}$$

$$\boxed{a = 1} \qquad \boxed{b = -\frac{10}{9}} \qquad \boxed{c = \frac{1}{9}}$$

Exercise 12 - continued

Inverse transformation

$$F(s) = \frac{1}{s} - \frac{10}{9}\frac{1}{s+1} + \frac{1}{9}\frac{1}{s+10}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$f(t) = 1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}$$

final value $\rightarrow$ $f(t)\Big|_{t\to\infty} = 1 - 0 + 0 = 1$ //

(same as before)

Exercise 13

Consider the following differential equation in the frequency domain

$$F(s) = \frac{1}{s(s+2)^2}$$

Determine:

$\rightarrow$ The final value of the function $f(t)$ when $t \rightarrow \infty$

$\rightarrow$ The function $f(t)$

Exercise 13 - continued

Final value

$$F(s) = \frac{1}{s(s+2)^2}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left(\frac{\cancel{s}\cdot 1}{\cancel{s}(s+2)^2}\right) \rightarrow \lim_{s\to 0} \frac{1}{(s+2)^2} = \frac{1}{4} //$$

Exercise 13 - continued

$$\Rightarrow \frac{a}{s} + \frac{b}{s+2} + \frac{c}{(s+2)^2} = \frac{1}{s(s+2)^2}$$

Inverse transformation

from partial fraction

$$F(s) = \frac{1}{4s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

$$\boxed{f(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}}$$

see slide 21, line 6.

Exercise 14

Assuming all initial conditions are zero, determine the solution of the following differential equation, where the forcing term is $\boxed{f(t) = 2e^{-t}.}$

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = f(t)$$

$\mathcal{L} = \frac{2}{s+1}$

Home work

try this on your own.

Exercise 14 - continued

$$(s^3 + 4s^2 + 5s + 2)\,Y(s) = \frac{2}{s+1}$$

$$Y(s) = \frac{2}{(s+1)(s^3+4s^2+5s+2)} = \frac{2}{(s+1)^3(s+2)}$$

roots $-1, -1, -2$

partial fraction

$$Y(s) = \frac{a}{(s+1)^3} + \frac{b}{(s+1)^2} + \frac{c}{(s+1)} + \frac{d}{s+2}$$

$a = 2$

$b = -2$

$c = 2$

$d = -2$

Exercise 14 - continued

partial fraction

$$Y(s) = \frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{2}{s+1} - \frac{2}{s+2}$$

$$y(t) = t^2 e^{-t} - 2te^{-t} + 2e^{-2t} - 2e^{-2t}$$

Exercise 15

A laser printer uses a laser beam to print copy rapidly for a computer. The laser is positioned by a control input $r(t)$ so that we have:

(not 4)

$$Y(s) = \frac{6(s+50)}{s^2+40s+300} R(s).$$

The input $r(t)$ represents the desired position of the laser beam. If $\underbrace{r(t) = 1}_{\mathcal{L}}$, determine:

→ The output $y(t)$

→ The final value of $y(t)$

$R(s) = \frac{1}{s}$

Exercise 15 - continued

1 - Partial fraction expansion ⚠

$$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300} R(s). = \frac{6(s+50)}{(s+10)(s+30)} \frac{1}{s}$$

$$\frac{a}{s+10} + \frac{b}{s+30} + \frac{c}{s} \quad \rightarrow \quad a = 1, \; b = -1.2, \; c = 0.2$$

$$Y(s) = \frac{1}{s} - \frac{1.2}{s+10} + \frac{0.2}{s+30}$$

2 - Inverse transform

$$y(t) = \mathcal{L}^{-1}(Y(s)) = 1 + 0.2e^{-30t} - 1.2e^{-10t}$$

if $y \rightarrow \infty$, $f(\infty) = 1$

Exercise 15 - continued

2 - Final value of $y(t)$

$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300} R(s).$

$$y(\infty) = \lim_{s \to 0} \cancel{s} \left(\frac{6(s+50)}{(s+10)(s+30)} \cdot \frac{1}{\cancel{s}} \right) = \frac{6 \times 50}{10 \times 30} = 1$$

Next class...

- Transfer functions