MECE 3350U
Control Systems

Lecture 21
State Space Models

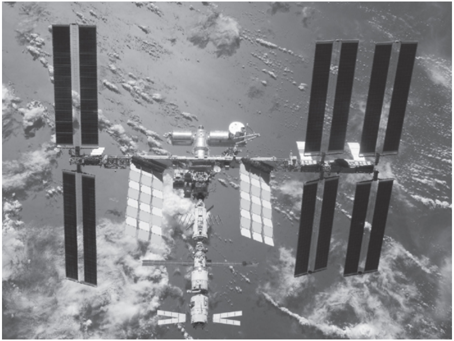
Outline of Lecture 21

By the end of today's lecture you should be able to

- Represent differential equations using matrices
- Obtain a state variable model for a given system
- Understand the role of state variables in design problems

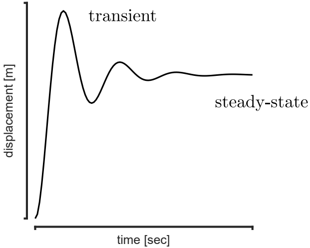
Applications

How can we determine the future orientation of the space station for a given control action?



Applications

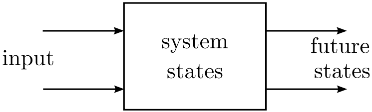
The step response of position controller of the surgical robotic arm is shown in the figure.



What parameters influence the transient and steady-state response?

Time domain and time invariant systems

State of a system: The set of variables that provides the future state and output of the system for a given input.

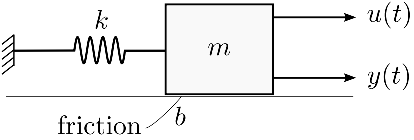


State variables: $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$

Examples: Position, velocity, voltage, current, etc.

Space state model

Consider the following system where $u(t)$ is the input and $y(t)$ is the output.

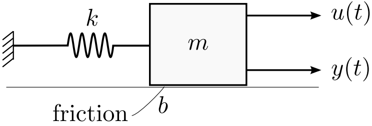


What variables do we need to know to predict the new state of the system when a force $u(t)$ is applied?

$$x(t) = [x_1(t),\ x_2(t)]$$

$$x_1(t) = \qquad\qquad , x_2(t) =$$

Space state model



State variables:

$$x_1(t) = y(t), \qquad x_2(t) = \frac{dy(t)}{dt}$$

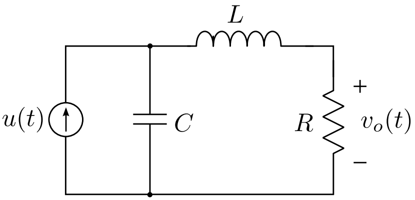
Differential equation describing the dynamic behaviour of the system

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$$

In terms of state variables:

Space state model

Consider the following system where $u(t)$ is the input and $v_o(t)$ is the output.

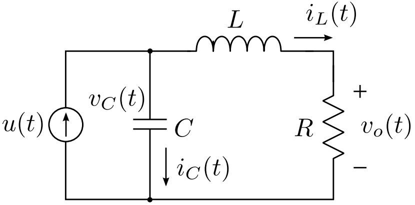


What variables do we need to know to predict the new state of the system when a current $u(t)$ is applied?

$$x(t) = [x_1(t),\ x_2(t)]$$

$$x_1(t) = \qquad\qquad , x_2(t) =$$

Space state model



$$x_1(t) = v_c(t), \qquad x_2(t) = i_L(t)$$

$$i_C(t) = C\frac{dv_c(t)}{dt} = u(t) - i_L(t)$$

$$L\frac{di_L}{dt} = -Ri_L(t) + v_c(t)$$

$$v_o(t) = Ri_L(t)$$

Space state model

$$x_1(t) = v_c(t), \qquad x_2(t) = i_L(t)$$

The state space equations are

$$\dot{x}_1(t) = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t)$$
$$\dot{x}_2(t) = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

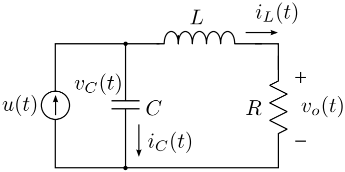
In matrix format we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u(t) \tag{1}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Bu(t)}$$

Output matrix

Recall that



$$\mathbf{x(t)} = [x_1(t) \quad x_2(t)]^T = [v_c(t) \quad i_L(t)]^T$$

$$y(t) = i_L(t)R = x_2(t)R$$

In the same way, the output signal can be written as

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u(t)}$$

$$y(t) = [0 \quad R]\mathbf{x}(t)$$

The space state representation is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u(t)}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u(t)}$$

State space - standard form

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}\dots & a_{1n} \\ a_{21} & a_{22}\dots & a_{2n} \\ \vdots & \dots & \vdots \\ a_{n1} & a_{n2}\dots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11}\dots & b_{1m} \\ \vdots & \vdots \\ b_{n1}\dots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Bu(t)}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{Du(t)}$$

Steps for analysis

1 $\rightarrow$ Derive the differential equations

2 $\rightarrow$ Define the state variables $x(t)$

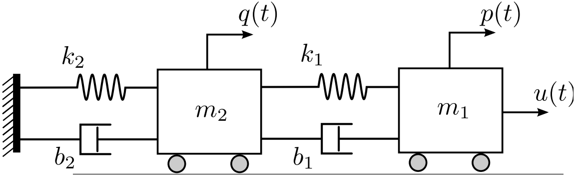
3 $\rightarrow$ Substitute step 2 in 1

4 $\rightarrow$ Arrange the equations in term of derivatives of $x(t)$

5 $\rightarrow$ Form the matrices for both state variables and outputs

Exercise 122

Find the state space equations of the rolling cart system. $q(t)$ and $p(t)$ denote the displacement of masses m_2 and m_1, respectively. $u(t)$ is the applied force. Choose the output to be $p(t)$.



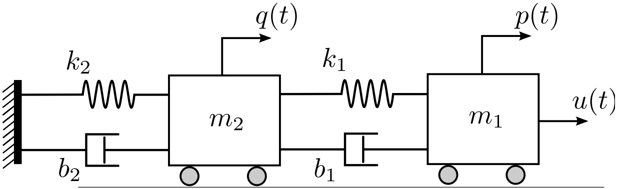
Procedure:

$\rightarrow$ Write the differential equations of motions for each cart

$\rightarrow$ Define the state variables and rework step 1

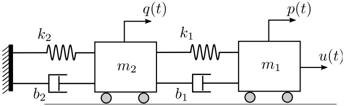
$\rightarrow$ Find the state space representation

Exercise 122 - continued



Step 1 - Differential equations of motion

Exercise 122 - continued



$$m_1\ddot{p}(t) + b_1\dot{p}(t) + k_1p(t) = u(t) + k_1q(t) + b_1\dot{q}(t)$$
$$m_2\ddot{q}(t) + (k_1 + k_2)q(t) + (b_1 + b_2)\dot{q}(t) = k_1p(t) + b_1\dot{p}(t)$$

Step 2 - Define the state variables

Step 3 - Differential equations with state variables

Exercise 122 - continued

$$
\begin{aligned}
\dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) \\
\ddot{x}_3(t) &= -\frac{k_1}{m_1}x_1(t) + \frac{k_1}{m_1}x_2(t) - \frac{b_1}{m_1}x_3(t) + \frac{b_1}{m_1}x_4(t) + \frac{1}{m_1}u(t) \\
\ddot{x}_4(t) &= \frac{k_1}{m_2}x_1(t) - \frac{k_1 + k_2}{m_2}x_2(t) + \frac{b_1}{m_2}x_3(t) - \frac{b_1 + b_2}{m_2}x_4(t)
\end{aligned}
$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Exercise 122 - continued

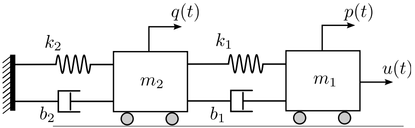
The output is $p(t)$.

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ q(t) \\ \dot{p}(t) \\ \dot{q}(t) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \quad\quad\quad\quad\quad\quad \end{bmatrix}, \quad \mathbf{D} = 0$$

Exercise 123

Draw the block diagram model of the state space equation of the two cart system from Exercise 1.



The state space model is

$$\dot{x}_1(t) = x_3(t)$$

$$\dot{x}_2(t) = x_4(t)$$

$$\dot{x}_3(t) = -\frac{k_1}{m_1}x_1(t) + \frac{k_1}{m_1}x_2(t) - \frac{b_1}{m_1}x_3(t) + \frac{b_1}{m_1}x_4(t) + \frac{1}{m_1}u(t)$$

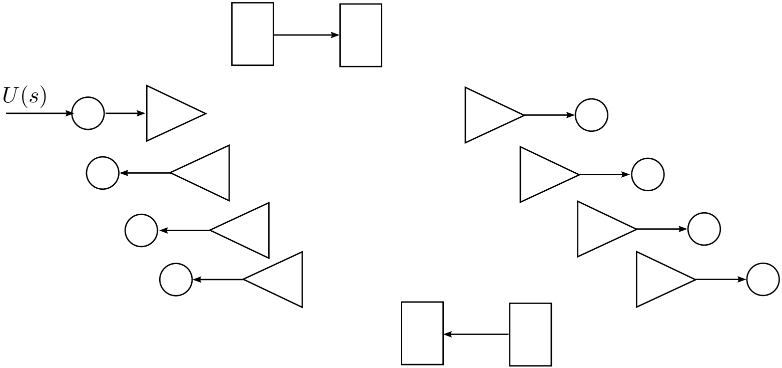
$$\dot{x}_4(t) = \frac{k_1}{m_2}x_1(t) - \frac{k_1 + k_2}{m_2}x_2(t) + \frac{b_1}{m_2}x_3(t) - \frac{b_1 + b_2}{m_2}x_4(t)$$

Exercise 123 - continued

$$\dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_4(t)$$

$$\dot{x}_3(t) = -\frac{k_1}{m_1}x_1(t) + \frac{k_1}{m_1}x_2(t) - \frac{b_1}{m_1}x_3(t) + \frac{b_1}{m_1}x_4(t) + \frac{1}{m_1}u(t)$$

$$\dot{x}_4(t) = \frac{k_1}{m_2}x_1(t) - \frac{k_1+k_2}{m_2}x_2(t) + \frac{b_1}{m_2}x_3(t) - \frac{b_1+b_2}{m_2}x_4(t)$$



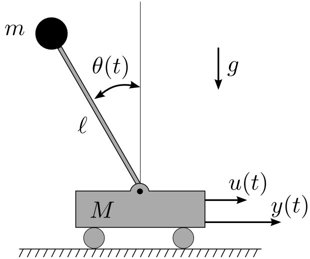
Exercise 124

The cart shown must be moved so that mass *m* is always in the upright position. The linearized equations of motion of the system are:

$$M\ddot{y}(t) + m\ell\ddot{\theta}(t) - u(t) = 0$$
$$ml\ddot{y}(t) + m\ell^2\ddot{\theta}(t) - m\ell g\theta(t) = 0$$

where $u(t)$ is the input force.

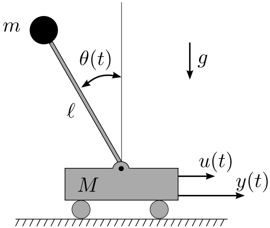
Find the state space equations and draw the corresponding block diagram.



Exercise 124 - continued

$$M\ddot{y}(t) + m\ell\ddot{\theta}(t) - u(t) = 0$$

$$ml\ddot{y}(t) + m\ell^2\ddot{\theta}(t) - m\ell g\theta(t) = 0$$



Exercise 124 - continued

$$M\ddot{y}(t) + m\ell\ddot{\theta}(t) - u(t) = 0$$

$$ml\ddot{y}(t) + m\ell^2\ddot{\theta}(t) - m\ell g\theta(t) = 0$$

Exercise 125

A system has the transfer function

$$\frac{Y(s)}{R(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}.$$

Construct a state variable representation of the system.

Procedure:

$\rightarrow$ Find the differential equation relating $y(t)$ and $r(t)$

$\rightarrow$ Define the state space variables

$\rightarrow$ Construct the state space matrices

Exercise 125 - continued

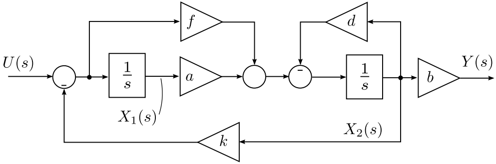
$$\frac{Y(s)}{R(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}.$$

Exercise 125 - continued

Exercise 126

A system is represented by the block diagram shown. Write the equations in the standard space state representation form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(\mathbf{t})$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(\mathbf{t})$$

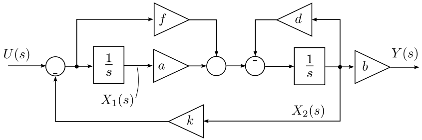


Procedure:

→ Find the equations for $x_1(t)$ and $x_2(t)$

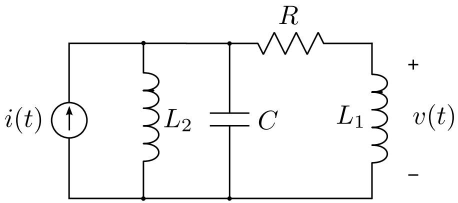
→ Construct the state space matrices

Exercise 126 - continued

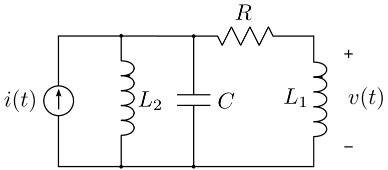


Exercise 127

Derive a state space model for the system shown. The input is $i(t)$ and the output is $v(t)$.



Exercise 127 - continued



Next class...

- Final exam review