MECE 3350U
Control Systems

Lecture 2
Dynamic Models

Outline of Lecture 2

By the end of this lecture you should be able to

- Model mechanical and electrical systems
- Find the differential equation that describes the behaviour of a physical system
- Understand the analogy between mechanical and electrical systems

Elements of a mechanical system

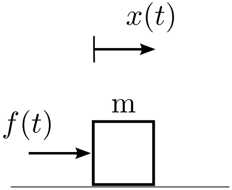
→ **Mass**: The quantity of matter in a body

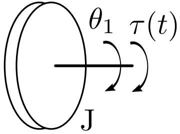
→ **Inertia**: Tendency to resist changes in state of motion

Idealization: Rigid body

$$f(t) = m\frac{d^2x}{dt} = m\ddot{x}$$

$$\tau(t) = J\frac{d^2\theta}{dt^2} = J\ddot{\theta}$$





Elements of a mechanical system

→ **Spring**: Designed to store energy

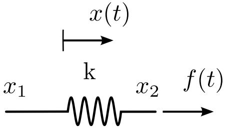
Idealization: Negligible mass and damping

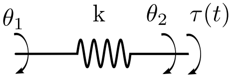
$$f(t) = K(x_2 - x_1)$$

$\hookrightarrow$ N/m

$$\tau(t) = K(\theta_2 - \theta_1)$$

$\hookrightarrow$ $\frac{Nm}{rad}$





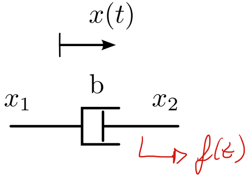
Elements of a mechanical system

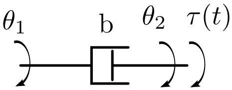
→ **Viscous damper**: Designed to dissipate energy

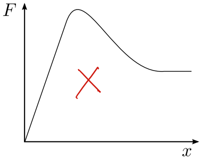
Idealization: negligible mass and stiffness

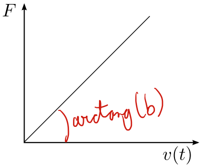
$$f(t) = b(\dot{x}_2 - \dot{x}_1)$$

$$\tau(t) = b(\dot{\theta}_2 - \dot{\theta}_1)$$





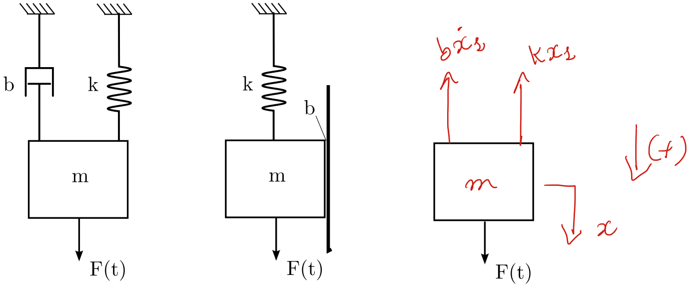




viscous friction
$\neq$
kinetic/static friction

Example

Find the equation of motion of the spring-mass-damper system.



$$\sum F = m\ddot{x}_1$$

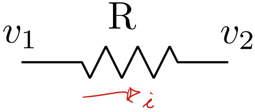
$$F(t) - b\dot{x}_1 - kx_1 = m\ddot{x}_1 \rightarrow \boxed{m\ddot{x}_1 + b\dot{x}_1 + kx_1 = f(t)}$$

Elements of electrical circuits

→ **Resistor**: Resistance against electric current

Idealization: No inductance or capacitance

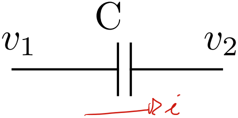
$$v = iR \quad , \quad i = \frac{v}{R}$$



→ **Capacitor**: Stores energy in an electric field

Idealization: No inductance or resistance

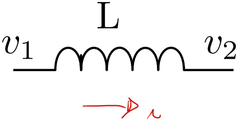
$$v = \frac{1}{C}\int i\,dt \quad , \quad i = C\frac{dv}{dt}$$



→ **Inductor**: Stores energy in a magnetic field

Idealization: No capacitance or resistance

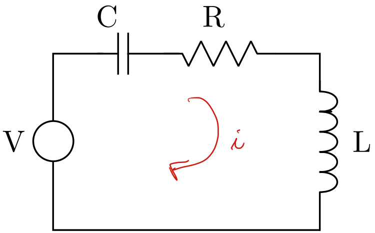
$$v = L\frac{di}{dt} \quad , \quad i = \frac{1}{L}\int i\,dt$$



Example

Find the relation between the voltage V, the current, and the charge in the circuit.

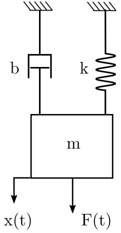
i q

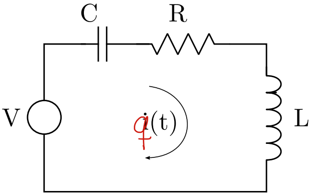


$$v = \frac{1}{C}\int i\,dt + iR + L\frac{di}{dt} \quad , \quad i = \frac{dq}{dt} \longrightarrow \text{electric charge.}$$

$$v = \frac{1}{C}q + \frac{dq}{dt}R + L\frac{d^2 i}{dt^2}$$

Mechanical/electrical analogy





$$F = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \tag{1}$$

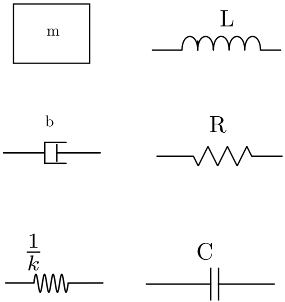
$$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \tag{2}$$

Impulse response

$$x(t) = Ke^{-\alpha t}\sin(\beta t + \theta) \tag{3}$$

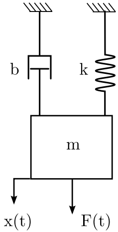
Mechanical/electrical analogy

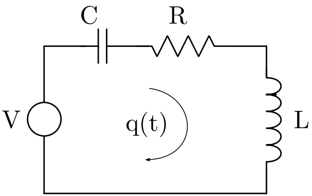
Mechanical		Electrical	
Force	F	Voltage	V
Velocity	v	Current	i
Displacement	x	Charge	q
Damping	b	Resistance	R
Mass	m	Inductance	L
Compliance	k^{-1}	Capacitance	C

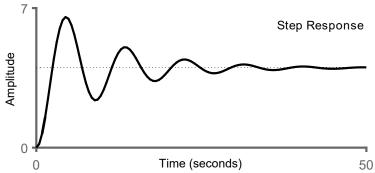


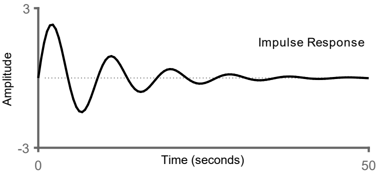
Mechanical/electrical analogy

Taking: $m = 5$ kg, $k = 0.25$ N/m, $b = 0.1$ Ns.









Mechanical/electrical analogy

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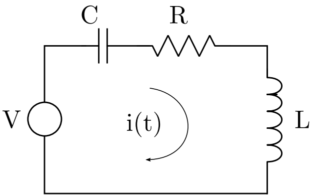
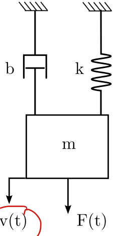
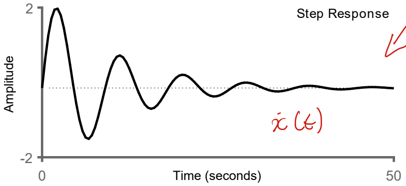
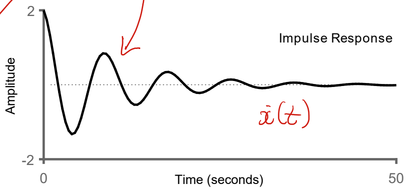
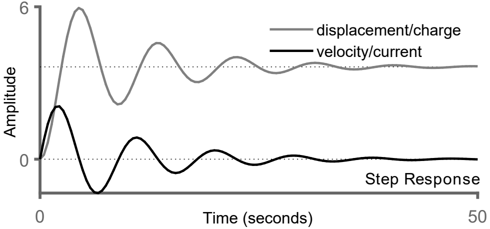


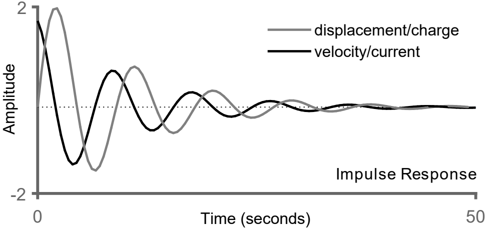






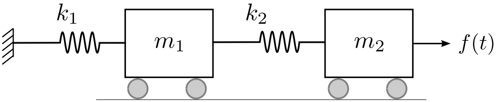
Mechanical/electrical analogy





Exercise 3

Find the equations of motion of the mass-spring system shown.

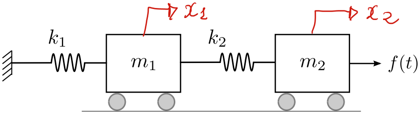


Procedure:

→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 3 - continued



Mass 1 $\quad \sum F = m\ddot{x}_1$

$$k_2(x_2-x_1) - k_1x_1 = m\ddot{x}_1$$

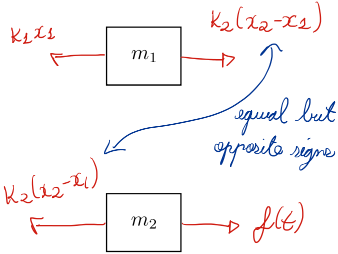
$$\boxed{m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0}$$

Mass 2 $\quad \sum F = m\ddot{x}_2$

$$m_2\ddot{x}_2 = f(t) - k_2(x_2-x_1)$$

$$\boxed{m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0} \rightarrow \mathcal{L}$$

$$m_2X_2(s)s^2 + k_2X_2(s) - k_2X_1(s) = 0$$

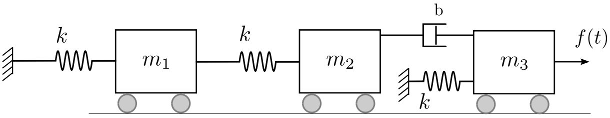


Exercise 4

Find the differential equations to model the behaviour of the system shown.

Home work

→ find the equation in the Laplace domain too.

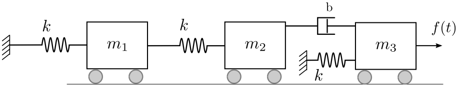


Procedure:

$\rightarrow$ Draw the free body diagram of each mass

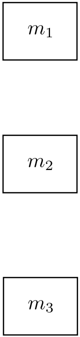
$\rightarrow$ Apply the equation of motion

Exercise 4 - continued



Homework

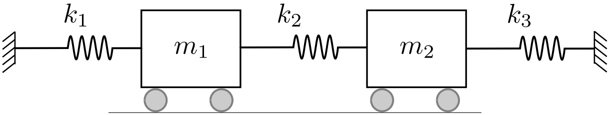
→ solution will follow.



Exercise 5

Find the differential equations to model the behaviour of the system shown.

Homework

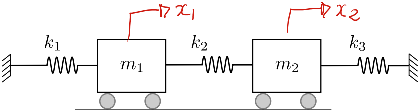


Procedure:

→ Draw the free body diagram of each mass

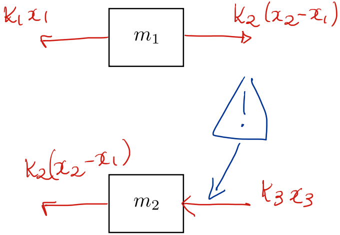
→ Apply the equation of motion

Exercise 5 - continued



Mass 1

$$\boxed{k_2(x_2-x_1) - k_1x_1 = m_1\ddot{x}_1}$$

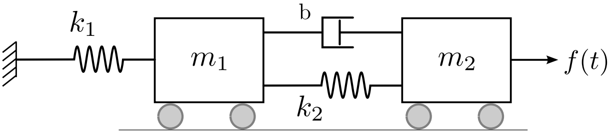


Mass 2

$$\boxed{-k_2(x_2-x_1) - k_3x_2 = m_2\ddot{x}_2}$$

Exercise 6

Find the differential equations to model the behaviour of the system shown.

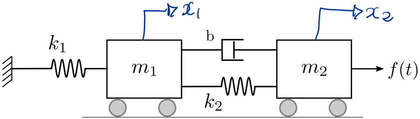


Procedure:

$\rightarrow$ Draw the free body diagram of each mass

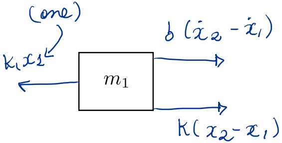
$\rightarrow$ Apply the equation of motion

Exercise 6 - continued



Mass 1

$$\boxed{b(\dot{x}_2-\dot{x}_1)+k(x_2-x_1)-k_1x_1=m_1\ddot{x}_1}$$

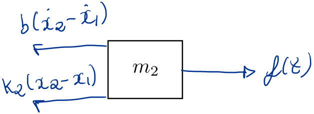


Mass 2

$$\boxed{f(t)-b(\dot{x}_2-\dot{x}_1)-k_2(x_2-x_1)=m_2\ddot{x}_2}$$

$\downarrow \mathcal{L}$

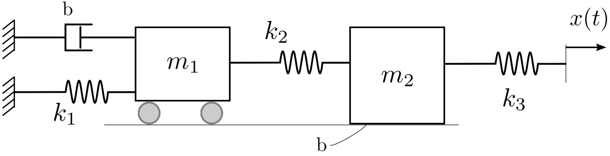
$$F(s)-b\left[X_2(s)-X_1(s)\right]s-k_2\left[X_2(s)-X_1(s)\right]=m_2X_2(s)s^2$$



Exercise 7

Find the differential equations to model the behaviour of the system shown.

Homework

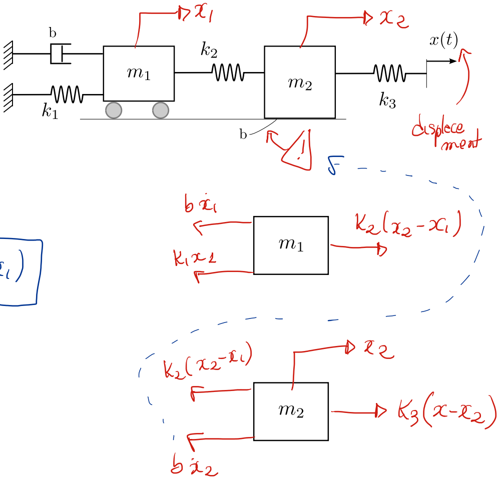


Procedure:

→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 7 - continued

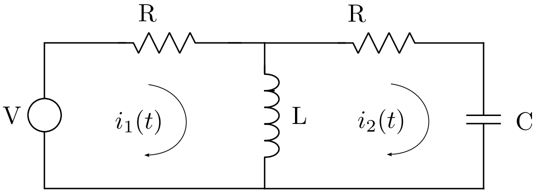


$$\boxed{m_1\ddot{x}_1 = K(x_2-x_1) - b\dot{x}_1 - Kx_1}$$

$$\boxed{m_2\ddot{x}_2 = K_3(x-x_2) - b\dot{x}_2 - K_2(x_2-x_1)}$$

Exercise 8

Write the the differential equations ($i = f(V)$) of the following circuit.

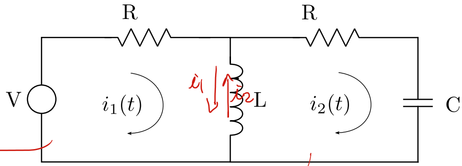


Procedure:

$\rightarrow$ Apply Kirchhoff's voltage law

$\rightarrow$ Find the equations for i_1 and i_2

Exercise 8 - continued



$$\boxed{V = i_1 R + L\frac{d}{dt}\left(i_1 - i_2\right)}$$

$$\left(V(s) = I_1(s)R + Ls\left(I_1 - I_2\right)\right)$$

Laplace

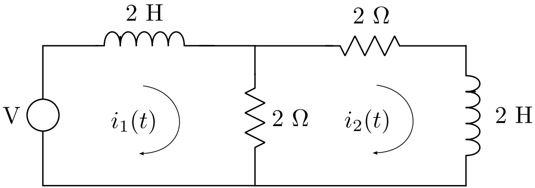
$$\boxed{i_2 R + \frac{1}{C}\int i_2\,dt + L\frac{d}{dt}\left(i_1 - i_2\right) = 0}$$

Laplace.

$$I_2(s)R + \frac{1}{Cs}I_2(s) + Ls\left(I_1(s) - I_2(s)\right)$$

Exercise 9

Write the the differential equations ($i = f(V)$) of the following circuit.

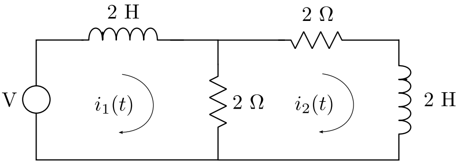


Procedure:

→ Apply Kirchhoff's voltage law

→ Find the equations for i_1 and i_2

Exercise 9 - continued

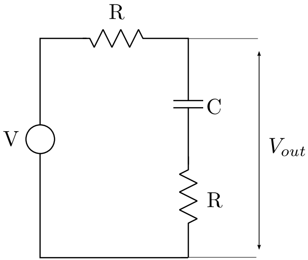


$$V = 2\frac{di_1}{dt} + 2(i_1 - i_2)$$

$$2(i_2 - i_1) + 2i_2 + 2\frac{di_2}{dt} = 0$$

Exercise 10

Write the the differential equations ($V_{out} = f(V)$) of the following circuit.

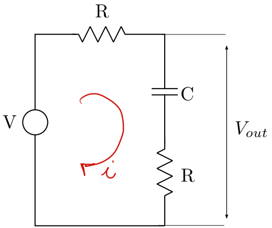


Procedure:

$\rightarrow$ Apply Kirchhoff's law

$\rightarrow$ Find the equations for V_{out} as a function of V

Exercise 10 - continued



$$-V + iR + \frac{1}{C}\int i\,dt + iR = 0$$

$$\boxed{V = 2iR + \frac{1}{C}\int i\,dt} \to \mathcal{L}$$

$$\left(V(s) = 2I(s)R + \frac{1}{Cs} I(s) \right)$$ Laplace domain

$$\boxed{V_{out} = \frac{1}{C}\int i\,dt + iR.}$$

Next class...

- Laplace transform