MECE 3350U
Control Systems

Lecture 19

Nyquist Plots

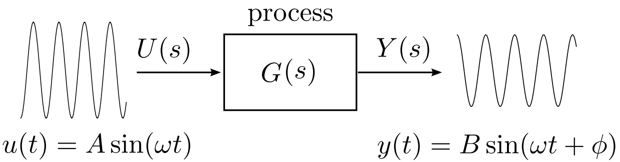
Outline of Lecture 19

By the end of today's lecture you should be able to

- Draw the approximate Nyquist plot of a transfer function
- Relate the Nyquist plot to frequency response
- Determine the stability based on open look transfer function

Applications

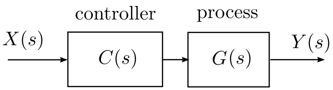
The frequency response of any system can be determined experimentally.



What does this information about the open-loop system tell us about the stability of the closed-loop system?

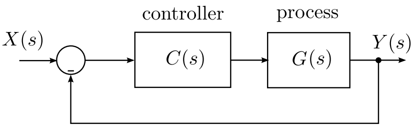
Review

An open loop transfer function $L(s) = C(s)H(s)$



is stable if all the **poles** of $C(s)H(s)$ have negative real parts.

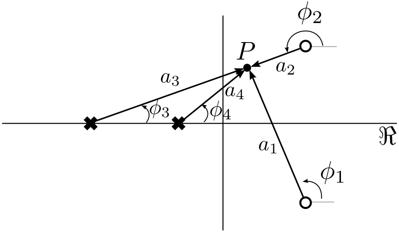
The closed loop system

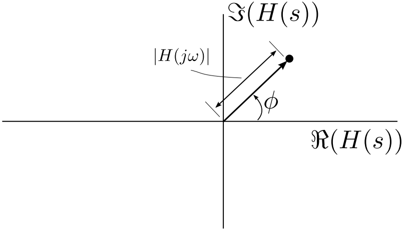


$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

is stable if the **zeros** of $1 + C(s)G(s)$ have negative real parts.

Cauchy's argument principle





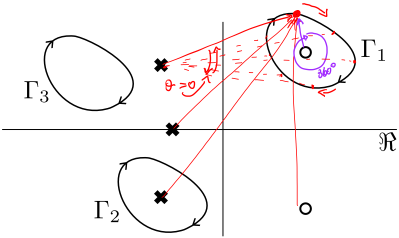
The magnitude is

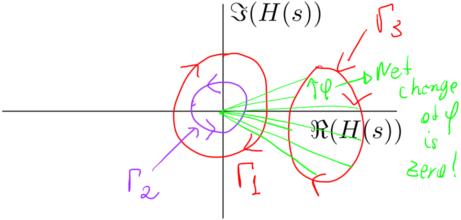
$$|H(j\omega)| = \frac{a_1 \times a_2}{a_3 \times a_4}$$

The phase is

$$\phi = \phi_1 + \phi_2 - \phi_3 - \phi_4$$

Cauchy's argument principle





As s traverses Γ_1, the net angle change is $+360^\circ$

As s traverses Γ_2, the net angle change is -360° (pole)

As s traverses Γ_3, the net angle change is 0

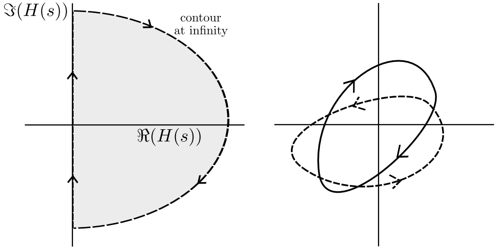
Cauchy's argument principle

If the characteristic equation of $1 + C(s)G(s)$ has:

→ A number P of **poles** in the right-half plane.

→ A number N of **zeros** in the right-half plane.

For an contour that encircles the entire right-half plane:



The relation between P, Z, and the **net** number N of clockwise encirclements of the origin is:

$$N = Z - P$$

$\boxed{Z = P + N}$

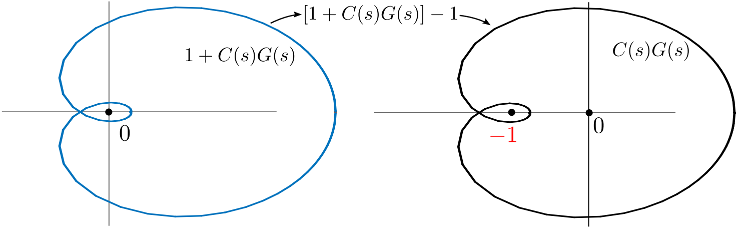
Nyquist plot

$$1 + C(s)G(s) = 0 \qquad \text{closed-loop} \tag{1}$$

If (1) has a zero or pole in the right-half s-plane, the contour of (1) encircles the origin.

$$T(s) = C(s)G(s) \qquad \text{open-loop} \tag{2}$$

If (2) has a zero or pole in the right-half s-plane, the contour of (1) encircles $-1 + j0$.



The Nyquist Stability Criterion

An **open-loop** transfer function $L(s)$ has Z unstable **closed-loop** roots given by

$$Z = N + P$$

Stability requires $Z = 0$.

→ N is the number of clockwise encirclements of -1

→ P is the number of poles in the right-half s-plane

Counterclockwise encirclements are negative.

Thus:

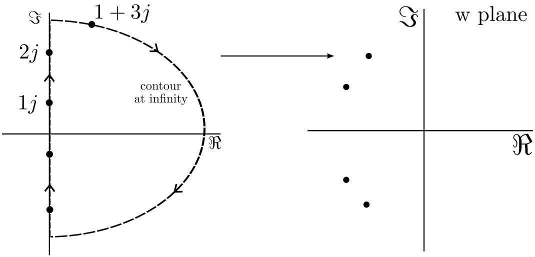
A **open-loop** transfer function $L(s)$ is **closed-loop** stable if and only if the number of counterclockwise encirclements of the $-1 + 0j$ point is equal to the number of poles of $L(s)$ with positive real parts.

Nyquist plot

How to create the Nyquist plot for a given function?

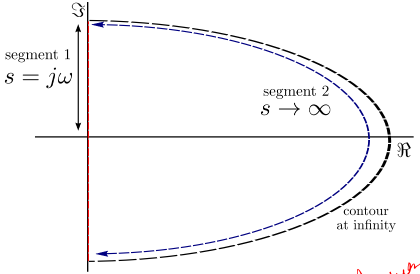
$$L(s) = \frac{s+1}{s^2+3}$$

Point by point mapping?



Nyquist plot

The Nyquist contour can be divided into two segments



$\Rightarrow$ Segment 1 - The imaginary axis, i.e., $s = j\omega$ → frequency response.

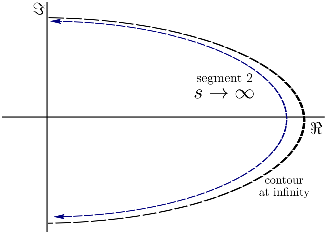
Thanks to symmetry, only the positive part needs to be evaluated

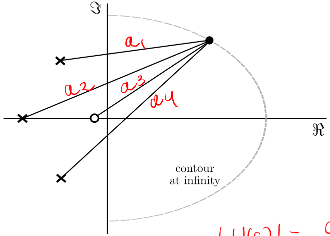
$\Rightarrow$ Segment 2 - The contour at infinity

Segment 2 maps to a single point!

Segment 2 - Contour at infinity

Case 1 - More poles than zeros





$|H(s)| = \dfrac{a_3}{a_2 a_1 a_4} \to 0$

If $|s| \to \infty$ and $m > n$, then

$$|H(s)| = k \frac{\prod_{i=1}^{n}(|s + z_i|)}{\prod_{k=1}^{m}(|s + p_k|)} \to 0$$

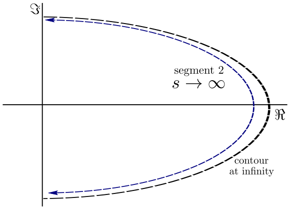
$\to$ The magnitude is zero for all points lying on the contour at infinity

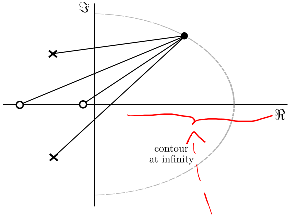
$\to$ The phase is irrelevant

$\to$ In the Nyquist plot the entire segment maps to zero.

Segment 2 - Contour at infinity

Case 2 - Same number of poles and zeros





If $|s| \to \infty$ and $m = n$, then

$$|H(s)| = k\frac{\prod_{i=1}^{n}(|s+z_i|)}{\prod_{k=1}^{m}(|s+p_k|)} = \beta$$

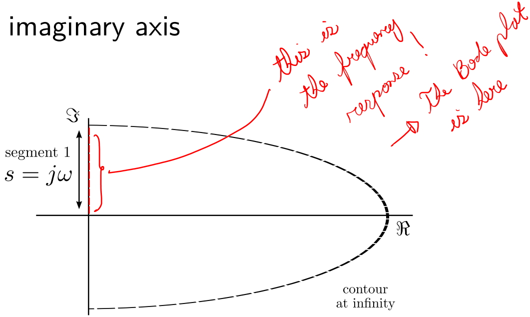
The phase is

$$\angle H(s) = \sum_{i=1}^{n}\angle(s+z_i) - \sum_{k=1}^{m}\angle(s+p_k) \approx 0$$

Thus

$$\boxed{\beta \in \Re^+}$$

Segment 1 - positive imaginary axis



For the imaginary segment, 4 points need to be analysed

$1 \rightarrow \omega = 0$ (starting point)

$2 \rightarrow \omega \rightarrow \infty$

$3 \rightarrow$ Point in the w-plane where the plot crosses the real axis

$4 \rightarrow$ Point in the w-plane where the plot crosses the imaginary axis

Exercise 112

recall that $\begin{cases} s = j\omega \\ j = \sqrt{-1} \end{cases}$

Determine the Nyquist plot for the open-loop transfer function

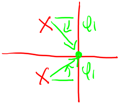
$$L(s) = \frac{1}{s^2 + s + 1} \rightarrow L(j\omega) = \frac{1}{(j\omega)^2 + j\omega + 1}$$

$$L(j\omega) = \frac{1}{-\omega^2 + j\omega + 1} \cdot \frac{(-\omega^2+1) - j\omega}{(-\omega^2+1) - j\omega} \rightarrow L(j\omega) = \underbrace{\frac{-\omega^2+1}{(-\omega^2+1)^2+\omega^2}}_{\text{real part}} + j\,\underbrace{\frac{-\omega}{(\omega^2+1)^2+\omega^2}}_{\text{imaginary part}}$$

① $\omega = 0$

$L(0) = 1$

phase is $-(\varphi_1 + \varphi_2)$

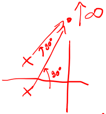


$\varphi_1 = -\varphi_2$

$\boxed{\varphi = 0}$

② $\omega \rightarrow \infty$

$L(\infty) = 0$



phase is $0 - (90° + 90°) = -180°$

pole

Imaginary axis crossing $Re = 0$

$$\frac{-\omega^2+1}{(-\omega^2+1)^2+\omega^2} = 0 \quad , \quad \omega = 1$$

replace $\omega = 1$ in $L(j\omega)$

$\boxed{L(j\omega) = \pm j}$

Real axis crossing $Im = 0$

$$\frac{-\omega}{(-\omega^2+1)^2+\omega^2} = 0 \quad , \quad \omega = 0 \rightarrow \text{already calculated in ①}$$

Exercise 112 - continued

$$L(s) = \frac{1}{s^2 + s + 1} \rightarrow L(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

$\omega = 0$ $\omega \rightarrow \infty$

Real axis crossing

See previous slide

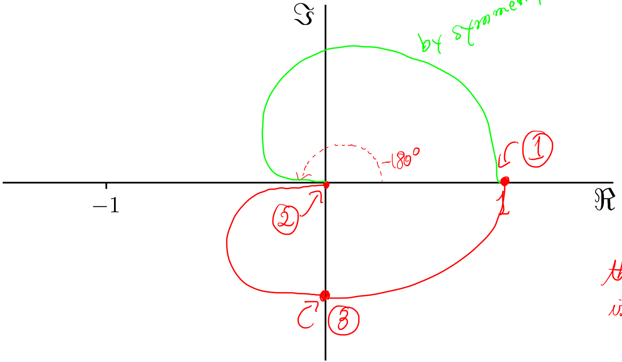
Imaginary axis crossing

Exercise 112 - continued

$\frac{1}{s^2+s+1}$

1. Starting point: $\omega = 0 \rightarrow w = 1\angle 0^\circ$
2. Mid point: $\omega = \infty \rightarrow w = 0\angle -180^\circ$
3. Imaginary axis crossing point: $w = \pm 1j$
4. Real axis crossing point: $w = 0$

same as 1

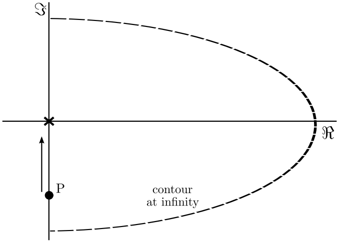


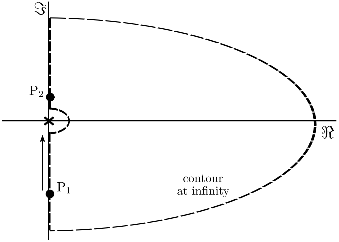
this system is always stable

Poles or zeros on the imaginary axis

A pole or zero anywhere on the imaginary axis will create an arc at infinity.

Example: $H(s) = \frac{1}{s}$





As P tends to zero:

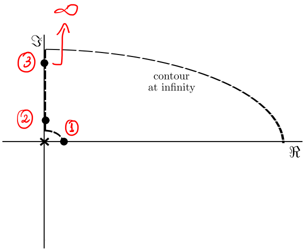
$|H(j\omega)| \to \infty$ and $\angle H(j\omega) = -\angle s = 0 - (-90) = 90^\circ$ but it is **undefined at** 0

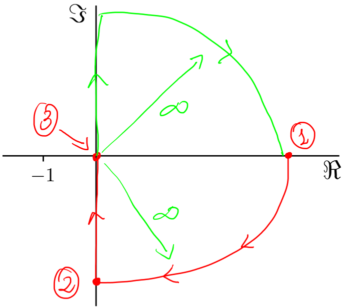
As P_1 follows the contour **around** 0

$|H(j\omega)| \to \infty$ and $\angle H(j\omega) = +90^\circ$, $\angle H(j\omega) = 0^\circ$, $\angle H(j\omega) = -90^\circ$

Poles or zeros on the imaginary axis

Example: $H(s) = \frac{1}{s}$





① $|H(j\omega)| = \infty$

$\varphi = 0$

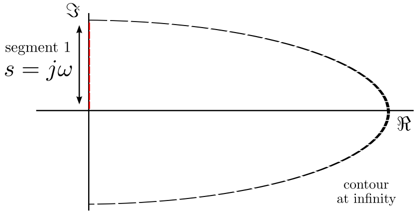
② $|H(j\omega)| = \infty$

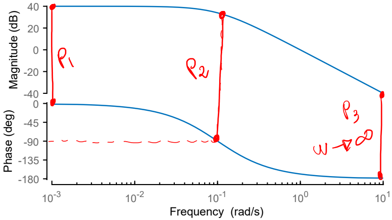
$\varphi = -90$ ⚠

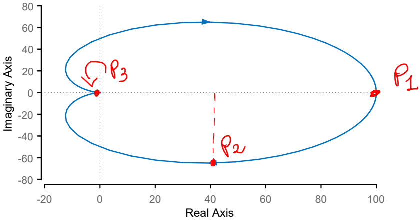
③ $|H(j\omega)| = 0$

$\varphi = -90$

Nyquist plot **vs** Bode plot







Steps for analysis

1 $\rightarrow$ In the transfer function, set $s = j\omega$

2 $\rightarrow$ Evaluate the points $\omega = 0$, and $\omega \rightarrow \infty$ (including phase)

3 $\rightarrow$ Find the points where the plot crosses the imaginary and real axis

4 $\rightarrow$ Sketch the Nyquist plot and draw the reflection about the real axis

5 $\rightarrow$ Evaluate the number N of clockwise encirclements of -1. If encirclements are in counterclockwise direction, N is negative.

6 $\rightarrow$ Determine the number P of unstable poles of the open-loop transfer function

7 $\rightarrow$ Calculate the number Z of unstable roots $Z = N + P$.

Exercise 113

Using the Nyquist stability criterion, evaluate the stability of a closed-loop system whose loop transfer function is

$$H(s) = \frac{1}{s(s+a)}$$

Exercise 113 - continued

$$H(s) = \frac{1}{s(s+a)}$$

$$H(j\omega) = \frac{1}{j\omega(j\omega+a)} = \frac{1}{-\omega^2+j\omega a}\,\frac{-\omega^2-j\omega a}{-\omega^2-j\omega a} \rightarrow H(j\omega) = \frac{-\omega^2}{\omega^2(\omega^2+a^2)} + j\,\frac{-\omega a}{\omega(\omega^3+\omega a^2)}$$

$$H(j\omega) = \frac{-1}{\omega^2+a^2} + j\,\frac{(-a)}{\omega^3+\omega a^2}$$

if $\omega = 0$

$$\boxed{H(j\omega) = -\frac{1}{a^2} + j(-\infty)}$$

Eq (1)

if $\omega \rightarrow \infty$

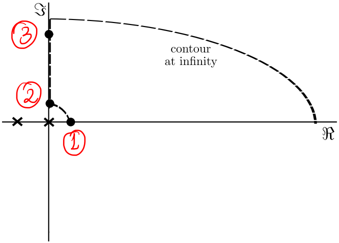
$$H(j\omega) = 0$$

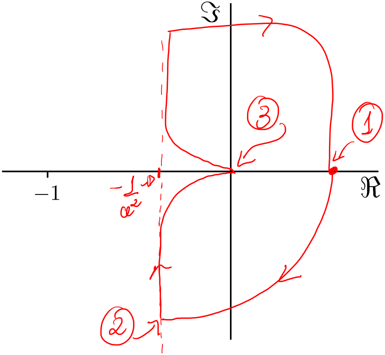
Exercise 113 - continued

1 $|H(j\omega)| = \infty$
$\phi = 0$

$$H(s) = \frac{1}{s(s+a)}$$

2 $|H(j\omega)| = \infty$ (same as Eg 1)
$\phi = -90^\circ$





3 $|H(j\omega)| = 0$
$\phi = -180^\circ$

Exercise 114

A closed-loop system has a loop transfer function

$$L(s) = k\frac{s+2}{s^2-1}.$$

Determine the minimum gain k that stabilizes the closed-loop system.

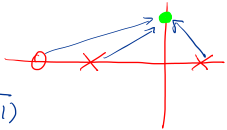
Use the Nyquist stability criterion.

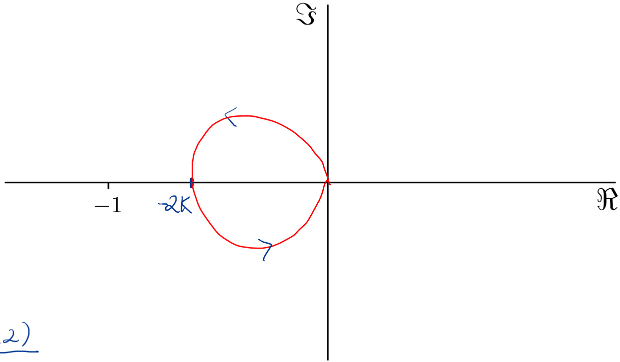
Exercise 114 - continued

$\omega = 0 \rightarrow -2K\angle 0^\circ$

$\omega \rightarrow \infty \rightarrow 0\angle -90^\circ$

$$L(s) = k\frac{s+2}{s^2-1} = \frac{s+2}{(s+1)(s-1)}$$





$$L(j\omega) = \frac{K(j\omega+2)}{(j\omega)^2-1}$$

$$L(j\omega) = \frac{2K}{-1-\omega^2} + j\frac{(K\omega)}{-1-\omega^2}$$

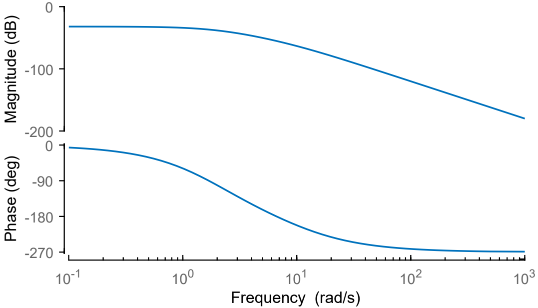
$Re = 0$, $\omega = \infty \rightarrow$ No crossing of Im.

$Im = 0$, $\omega = 0$ or $\omega = \infty$ (already calculated)

Exercise 115

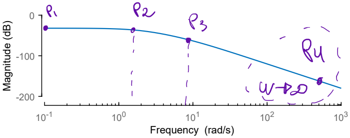
Sketch the Nyquist plot based on the Bode plots ($k = 1$) for the following system, then compare your result with that obtained using the Matlab command "nyquist". Using your plots, estimate the range of k for which the system is stable, and quantitatively verify your result using a rough sketch of a root-locus plot.

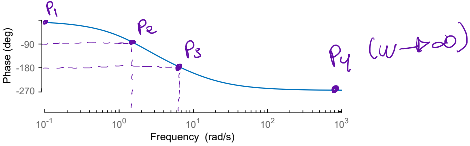
$$L(s) = \frac{k}{(s+10)(s+2)^2}$$



Exercise 115 - continued

$$L(s) = \frac{k}{(s+10)(s+2)^2}$$

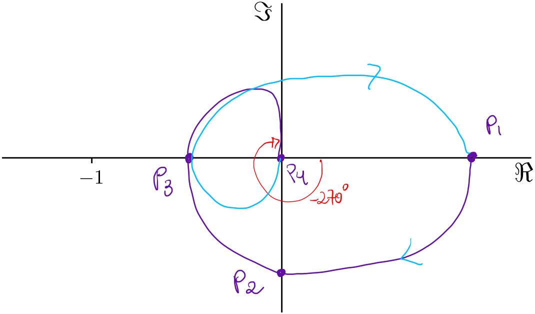




P4 $\omega \to \infty$

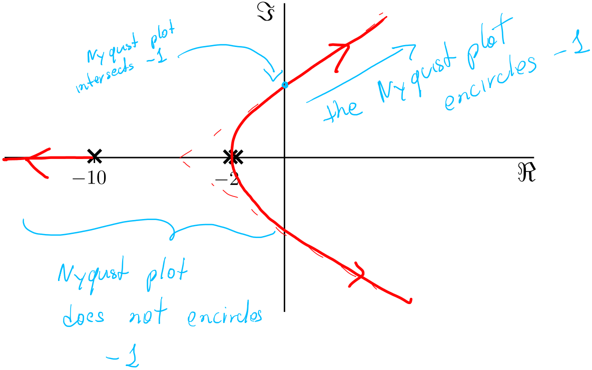
$|G(j\omega)| \to 0$

$\varphi = -270^\circ$



Exercise 115 - continued

$$L(s) = \frac{k}{(s+10)(s+2)^2}$$



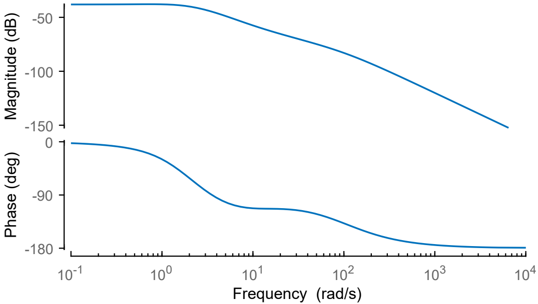
Exercise 116

Sketch the Nyquist plot based on the Bode plots ($k = 1$) for the following system, then compare your result with that obtained using the Matlab command "nyquist". Using your plots, estimate the range of k for which the system is stable, and quantitatively verify your result using a rough sketch of a root-locus plot.

$$L(s) = k\frac{(s+10)(s+1)}{(s+100)(s+2)^3}$$

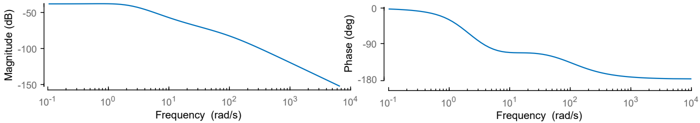
Homework.

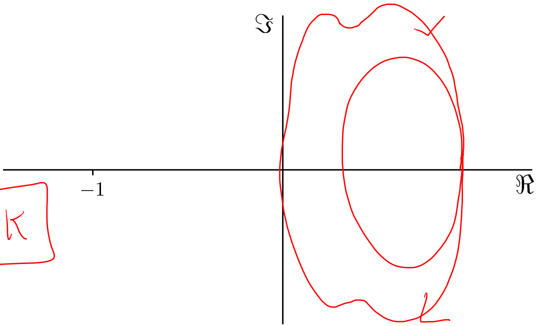
Check solutions with Matlab command "nyquist(H)"



Exercise 116 - continued

$$L(s) = k\frac{(s+10)(s+1)}{(s+100)(s+3)^2}$$

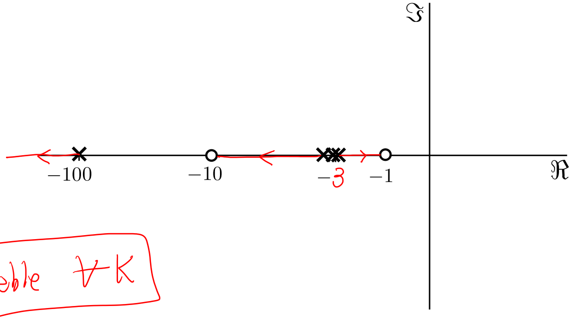




$\boxed{\text{stable } \forall K}$

Exercise 116 - continued

$$L(s) = k\frac{(s+10)(s+1)}{(s+100)(s+3)^2}$$



$\boxed{\text{stable } \forall K}$

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Next class...

- Stability margins