MECE 3350U
Control Systems

Lecture 18

Nyquist Stability Criterion

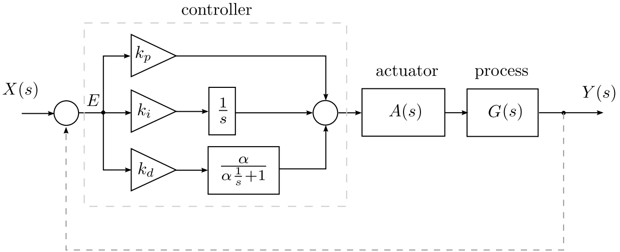
Outline of Lecture 18

By the end of today's lecture you should be able to

- Extend the concept of gain and phase
- Understand the Nyquist stability criterion
- Determine the stability based on open loop transfer function

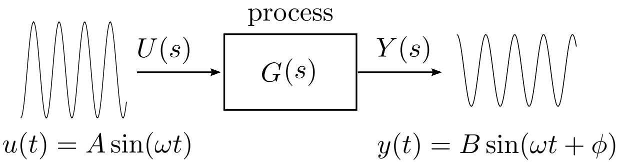
Applications

Knowing the open-loop transfer function of the system below, how can we evaluate its stability without computing the closed-loop transfer function?



Gain and phase - review

Any system can be characterized by its frequency response to a sinusoidal excitation.



The ratio B/A is called the gain of $G(s)$ for given frequency.

The phase shift ϕ is the phase of $G(s)$ for a given frequency.

Data can be obtained experimentally is $G(s)$ is unknown.

Gain and phase - review

For a generic transfer function $G(s)$

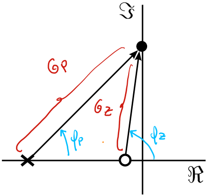
$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{k=1}^{m}(s+p_k)}$$

we can evaluate the **gain** at a frequency ω by letting $s = j\omega$.

The gain is

$$G(j\omega) = |k|\frac{\prod_{i=1}^{n}|j\omega+z_i|}{\prod_{k=1}^{m}|j\omega+p_k|}$$

where $|j\omega \pm a| = \sqrt{\omega^2 + (\pm a)^2}$



$G = \dfrac{G_z}{G_p}$

$\varphi = \varphi_z - \varphi_p$

Gain and phase - review

For a generic transfer function $G(s)$

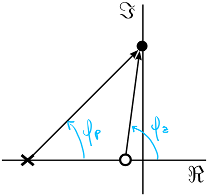
$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{k=1}^{m}(s+p_k)}$$

we can evaluate the **phase** at a frequency ω by letting $s = j\omega$.

The phase is

$$\angle G(j\omega) = \angle|k| + \sum_{i=1}^{n}\angle(j\omega+z_i) - \sum_{k=1}^{m}\angle(j\omega+p_k)$$

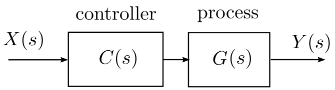
where $\angle(j\omega+a) = \tan^{-1}\omega/a$



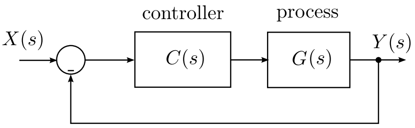
$\varphi = \varphi_p - \varphi_z$

Open loop *vs* closed loop stability

Generally the process and controller transfer functions are known



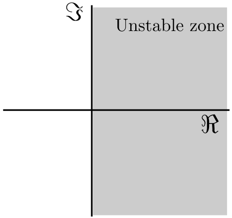
The open loop transfer function is $L(s) = C(s)H(s)$.



Closing the loop changes the transfer function to

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{L(s)}{1 + L(s)}$$

Open loop *vs* closed loop stability



Open-loop stability

$$T(s) = C(s)G(s)$$

$\rightarrow$ Evaluate the location of the **poles** of $C(s)G(s)$

Closed-loop stability

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

$\rightarrow$ Evaluate the location of the **zeros** of $1 + C(s)G(s)$

Example: If $C(s)G(s) = \frac{s+a}{s+b}$

$\rightarrow$ Open-loop stable if $C(s)G(s)$ has real negative **poles**: i.e., $b > 0$

$\rightarrow$ Closed-loop stable if $1 + C(s)G(s)$ has real negative **zeros**:

$$T(s) = \frac{\frac{s+a}{s+b}}{1 + \frac{s+a}{s+b}} = \frac{\frac{s+a}{s+b}}{\frac{(s+b)+(s+a)}{s+b}}$$

Open loop *vs* closed loop stability

The open loop transfer function

$$C(s)G(s) = \frac{s+a}{s+b} \tag{2}$$

has a zero at $-a$ and pole at $-b$.

The characteristic equation of the closed-loop transfer function in a unit feedback system becomes

$$1 + C(s)G(s) = 1 + \frac{s+a}{s+b} = \frac{s+a+s+b}{s+b} \tag{3}$$

and has a pole at $-b$.

The pole is (3) the same as in (2) !

For close-loop stability, the **zeros** of the characteristic equation, i.e. the zeros of $1 + C(s)G(s)$, must have negative real parts.

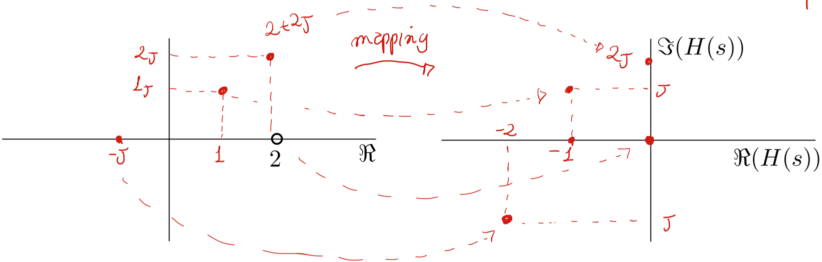
Function mapping

Consider the hypothetical function

$$H(s) = s - 2$$

How can we determine the location of the zeros of $H(s)$ graphically?

w-plane

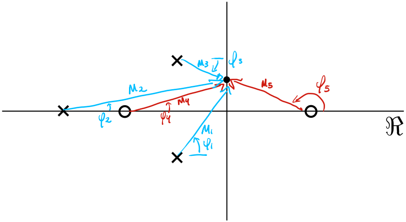


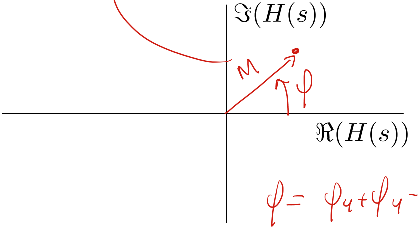
We can map any point from the s-plane into the "w" plane:

- $\rightarrow s = 2 + 2j$ becomes $H(2 + 2j) = 2j$
- $\rightarrow s = 1 + j$ becomes $H(1 + j) = -1 + j$
- $\rightarrow s = -j$ becomes $H(-j) = -2 - j$

Cauchy's argument principle

$$M = \frac{M_4 M_5}{M_1 M_2 M_3}$$



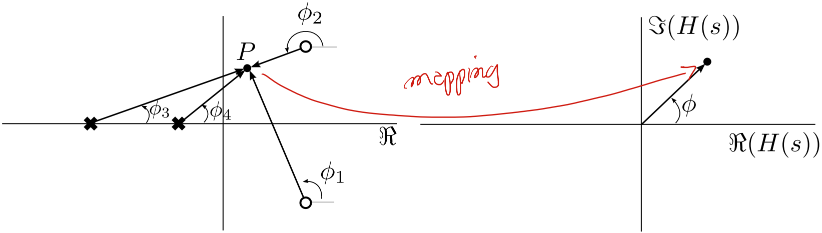


$$\varphi = \varphi_4 + \varphi_4 - \varphi_1 - \varphi_2 - \varphi_3$$

$$\begin{aligned}
H(s) =& |H(s)| \angle H(s) \\
=& |k| \frac{\prod_{i=1}^{n} |s+z_i|}{\prod_{k=1}^{m} |s+p_k|} \left(\sum_{i=1}^{n} \angle(s+z_i) - \sum_{k=1}^{m} \angle(s+p_k) \right) \\
=& |H(s)| \left(\sum \phi_i - \sum \phi_k \right)
\end{aligned}$$

Cauchy's argument principle

w-plane



1 - Select a point P in the s-plane

2 - Draw the vectors from P to each zero and pole

3 - Calculate the magnitude of each vector

4 - The magnitude is the product of magnitude of zeros divided by the product of the magnitude of poles

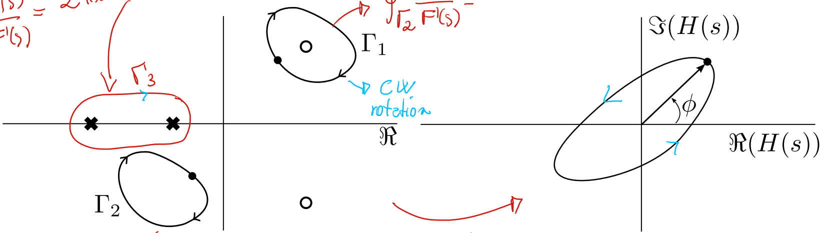
5 - The angle is

$$\phi = \phi_1 + \phi_2 - \phi_3 - \phi_4$$

Cauchy's argument principle → $\oint_{\Gamma} \frac{F(s)}{F'(s)} ds = 2\pi i \, (Z - P)$

$\oint_{\Gamma_3} \frac{F(s)}{F'(s)} = 2\pi i \, (2)$

$\oint_{\Gamma_2} \frac{F(s)}{F'(s)} = 2\pi i \, (1-0)$



$\oint_{\Gamma_2} \frac{F(s)}{F'(s)} = 0$

mapping

As s traverses Γ_2, the net angle change of ϕ is 0

As s traverses Γ_1, the net angle change of $|\phi|$ is $+360^\circ$

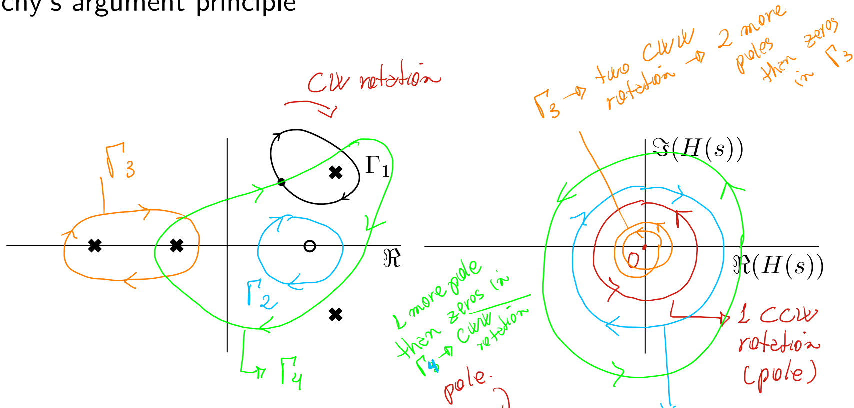
$\Gamma_3 \rightarrow -2 \times 360^\circ = 720^\circ$

the corresponding contour in the w-plane will encircle the origin $Z - P$ times.

$Z \rightarrow$ # of zeros in the contour
$P \rightarrow$ # of poles in the closed integral

DEMOSTRATION:
See Matlab code posted on BB

Cauchy's argument principle



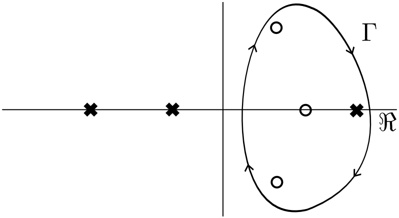
As s traverses Γ_1, the net angle change of ϕ is -360^0

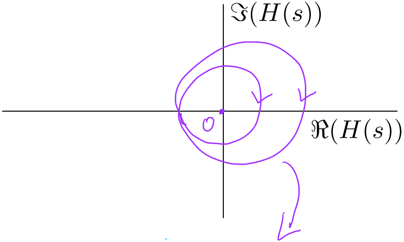
$\Gamma_2 \Rightarrow +360^0$

$\Gamma_3 \Rightarrow -720^0$

$\Gamma_4 \Rightarrow -360^0$

Cauchy's argument principle





As s traverses Γ_1, the net angle change of ϕ is

two encirclements of the origin, thus there are two MORE zeros than poles within the contour defined by Γ

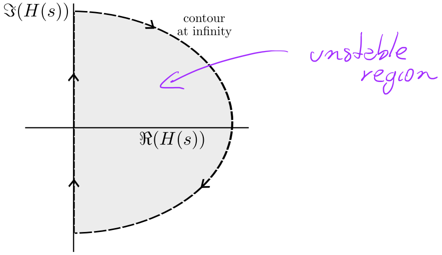
Cauchy's argument principle

Assume that the characteristic equation of $1 + C(s)G(s)$ has:

→ A number P of **poles** in the right-half plane.

→ A number N of **zeros** in the right-half plane.

For an contour that encircles the entire right-half plane:



The relation between P, Z, and the **net** number N of clockwise encirclements of the origin is:

$$\boxed{N = Z - P}$$

Cauchy's argument principle

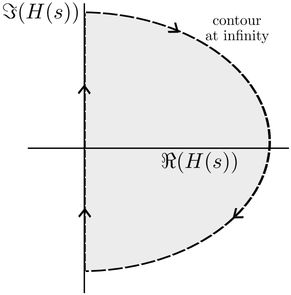
A contour map of a complex function will encircle the origin

$$\boxed{N = Z - P}$$

times, where Z is the number of zeros and P is the number of poles of the function inside the contour.

for stability

$Z = 0$



The number of unstable poles are known: They are the same as in the open-loop transfer function!

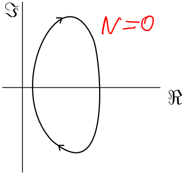
Nyquist plot

$P = 0$

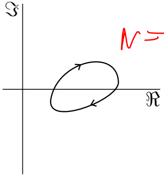
Assuming that there are no poles in the right-half plane, are the following systems stable?

$$1 + C(s)G(s) = 0$$

$Z = P + N$
$Z = 0$
stable!



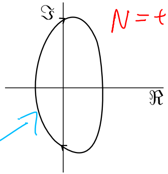
$N = 0$



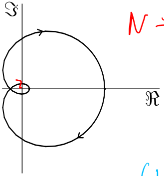
$N = 0$

$Z = P + N$
$Z = 0$
stable

$Z = P + N$
$Z = 0 + 1$
$Z = 1$
unstable!



$N = +1$



$N = +2$

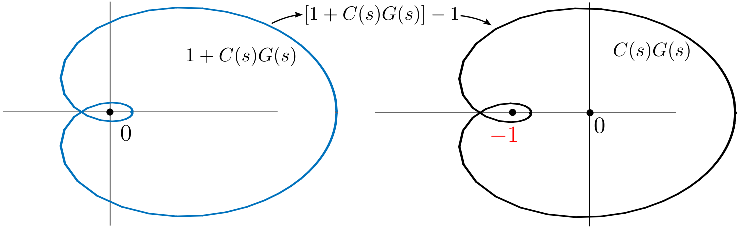
$Z = P + N$
$Z = 2$
unstable!
(two unstable zeros)

Nyquist plot

$$1 + C(s)G(s) = 0. \tag{4}$$

If there is a zero or pole of (4) in the right-half s-plane, the contour of (4) encircles the origin.

Subtracting 1 from the above equation shifts the contour to the left



Thus, if the **open-loop** equation

open loop
Transfer function

$$T(s) = C(s)G(s) \tag{5}$$

has a zero or pole in the right-half s-plane, the contour of (5) encircles **-1**.

The Nyquist Stability Criterion

An **open-loop** transfer function $L(s)$ has Z unstable **closed-loop** roots given by

$$\boxed{Z = N + P}$$

stability requires
$N = 0$

where

→ N is the number of clockwise encirclements of -1

→ P is the number of poles in the right-half s-plane

Note: If encirclements are in the counterclockwise direction, N is negative.

For stability, we wish to have $Z = 0$.

The Nyquist Stability Criterion

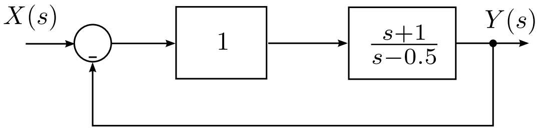
A open-loop transfer function $L(s)$ is closed-loop stable if and only if:

The number of counterclockwise encirclements of the $-1 + 0j$ point is equal to the number of poles of $L(s)$ with positive real parts.

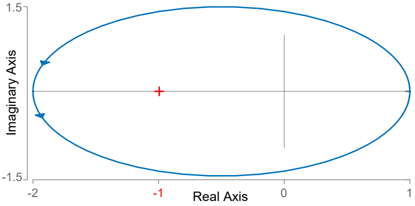
$$\boxed{Z = N + P}$$

Example 1

Is this closed-loop system stable?

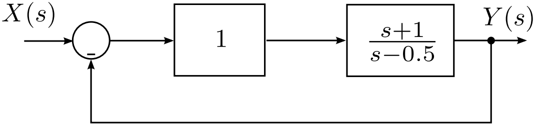


The Nyquist plot of the **open-loop** transfer function $L(s) = \frac{s+1}{s-0.5}$ is



$P = 1$ $(s = 0.5)$, $N = -1$ CCW , $Z = N + P = -1 + 1 = 0 \rightarrow$ stable!

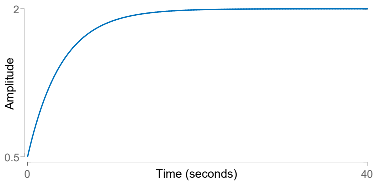
Example 1 - continued



The closed-loop transfer function is

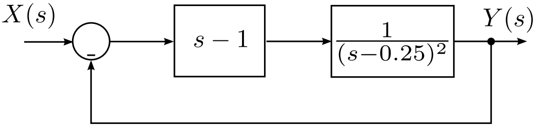
$$T(s) = \frac{\frac{s+1}{s-0.5}}{1+\frac{s+1}{s-0.5}} = \frac{s+1}{2s+0.5}$$

Step response

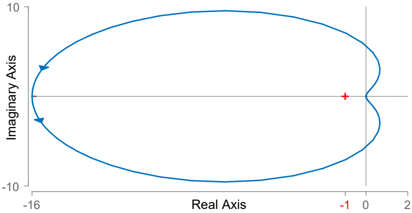


Example 2

Is this closed-loop system stable?



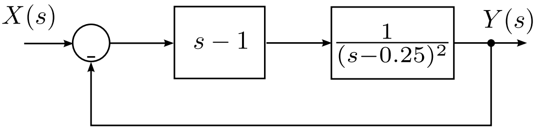
The Nyquist plot of the **open-loop** transfer function $L(s) = \frac{s-1}{(s-0.25)^2}$ is



$P =$ 2 , $N =$ -1 , $Z = N + P =$ -1+2 = 1 unstable!

(s=0.25
s=0.25)

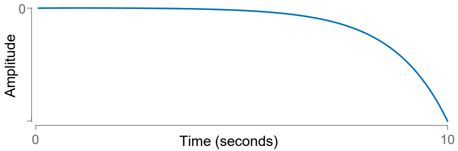
Example 2 - continued



The closed-loop transfer function is

$$T(s) = \frac{\frac{s-1}{s^2-0.5s+0.0625}}{1+\frac{s-1}{s^2-0.5s+0.0625}} = \frac{s-1}{s^2+0.5s-0.9375}$$

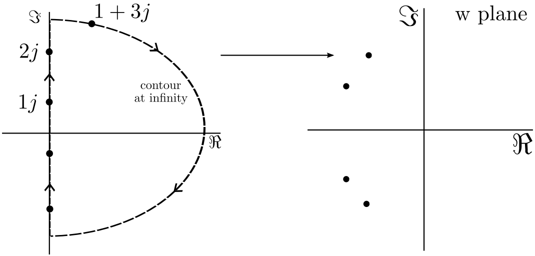
The poles are: -1.25 and 0.75 and the step response is



$1 + L(s)$ has ONE unstable **zero**.

Nyquist plot

How to create the Nyquist plot for a given function?

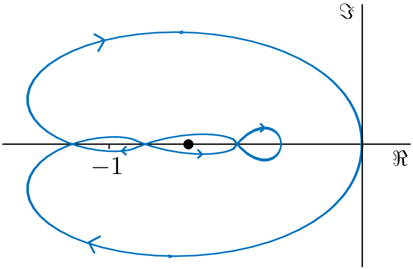


Next class!

Exercise 105

The Nyquist plot of a conditionally stable open loop system is shown in the figure.

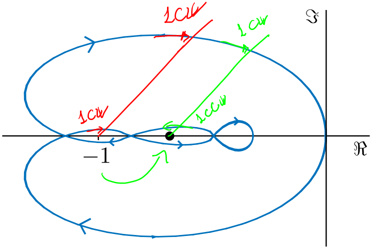
$\hookrightarrow P=0$



(a) Determine whether the closed-loop system is stable

(b) Determine whether the closed-loop system is stable if the -1 point lies at the dot on the axis.

Exercise 105 - continued



(a)

$N = 2$

$P = 0$

$Z = P + N$

$\boxed{Z = 2}$ unstable!

1CW
- 1CCW rotation

(b)

$N = 0$

$P = 0$

$Z = P + N$

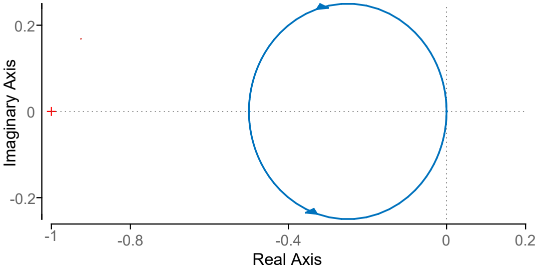
$\boxed{Z = 0}$ stable!

Exercise 106

A unit feedback system has a loop transfer function

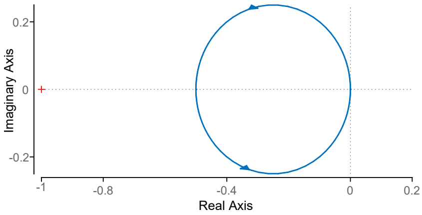
$$L(s) = C(s)G(s) = \frac{k}{\tau s - 1}$$

where $k = 0.5$ and $\tau = 1$. Based on its Nyquist plot show below, determine whether the system is stable.



Exercise 106 - continued

$$L(s) = C(s)G(s) = \frac{0.5}{s-1}$$



$P = 1$

$N = 0$ (no encirclements)

$Z = P + N$

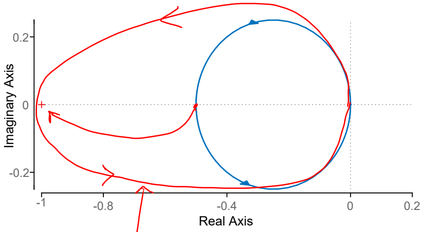
$Z = 1$

$\hookrightarrow$ 1 unstabe closed-loop pole!

Exercise 106 - continued

What value of k is required for stability?

$$L(s) = C(s)G(s) = \frac{k}{s-1}$$



$L(s) = \frac{1}{s-1}$

Need to multiply the loop by 2

or $k = 1$

$N = -1$ (cww)

$Z = P + N$

$\boxed{Z = 1 - 1 = 0 \quad \text{stable}}$

proof $\Rightarrow$ $1 + \frac{K}{s-1} = 0$

$s - 1 + K = 0$

$-1 + K > 0$

if $\boxed{K > 1}$ (stable)

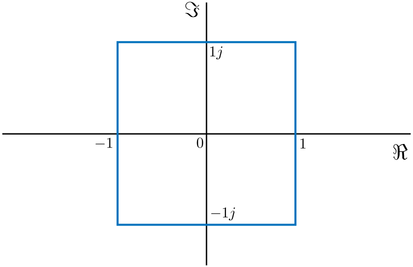
Exercise 107

home work

A loop transfer function is

$$L(s) = \frac{1}{s+2}$$

Using the contour in the s-plane shown, determine the corresponding contour in the $F(s)$ (or "w") plane.



Exercise 107 - continued

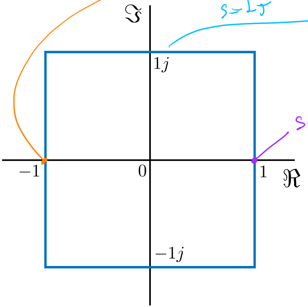
$s=-1$

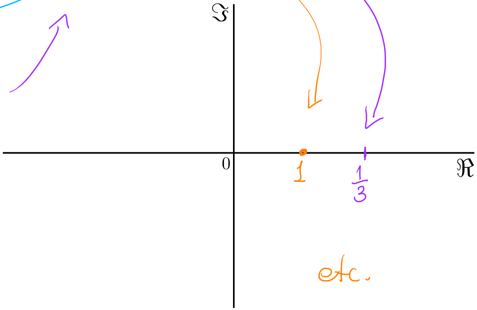
$s=1j$

$$L(s)=\frac{1}{s+2}$$

$\frac{1}{s+2} \times \frac{s-2}{s-2}$

$s=1$



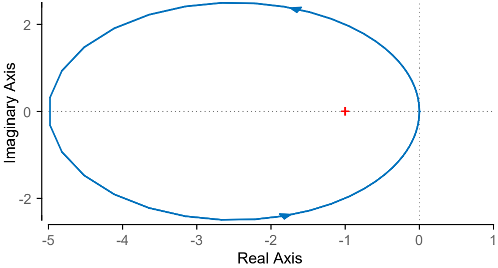


etc.

Exercise 108

Based on the Nyquist plot, evaluate the stability of

$$T(s) = \frac{s}{(s-0.1)^2}$$



$P = +2$

$N = -1$

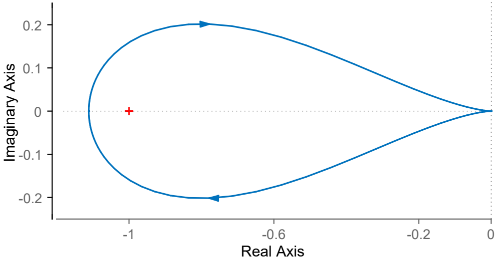
$Z = P + N = 1$

unstable!

Exercise 109

Based on the Nyquist plot, evaluate the stability of

$$T(s) = \frac{50}{(s+5)(s-9)}$$



$P = 1$

$N = 1$ (cw)

$Z = P + N = 2$

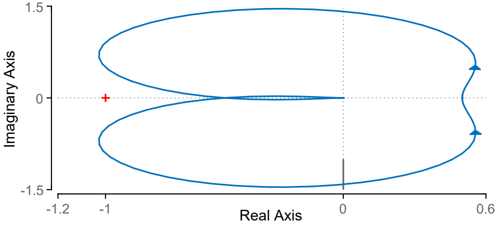
unstable!

Exercise 110

Based on the Nyquist plot, evaluate the stability of

$$T(s) = \frac{s + 0.5}{s^3 + s^2 + 1}$$

poles are
-1.46, $0.23 \pm 0.79i$
(2 unstable poles)



$P = 2$

$N = 0$

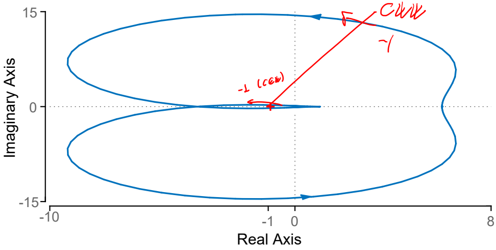
$Z = 2 + 0 = 2$

unstable!

Exercise 111

Based on the Nyquist plot, evaluate the stability of

$$T(s) = 10\frac{s+0.5}{s^3+s^2+1}$$



$P = 2$

$N = -2$

(2 cww encirc)

$\boxed{Z = 0}$

stable.

yay!

Next class...

- Nyquist plot