MECE 3350U
Control Systems

Lecture 16
Bode Plots 1/2

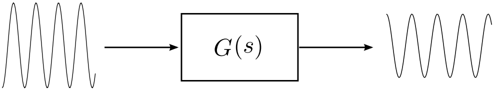
Outline of Lecture 16

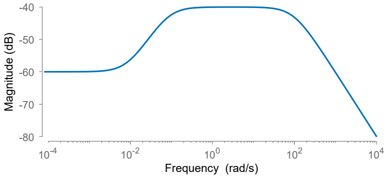
By the end of today's lecture you should be able to

- Understand the concept of frequency response
- Determine the magnitude of a transfer function
- Determine the phase of a transfer function
- Represent magnitude and phase in a Bode plot

Applications

The response of a unknown dynamic system to a sinusoidal excitation of variable frequency was measured using an oscilloscope.





What can we infer about the system ?

Applications

An object free to vibrate tends to do so at a specific rate called the object's or resonant frequency.

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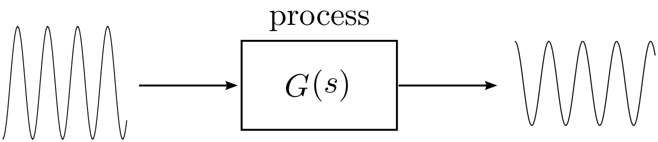
How can we identify the resonance frequency of a system?

Frequency response

Industrial design of control system is accomplished using frequency-response methods more often that any other.



Advantages:

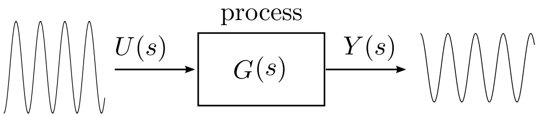
$\rightarrow$ Raw measurements of the output are sufficient for design

$\rightarrow$ Simple to compute

$\rightarrow$ No need to know the dynamic model of the system

Frequency response

Frequency response: The steady-state response of the system to a sinusoidal input signal.



Consider a system described by

$$\frac{Y(s)}{U(s)} = G(s) \tag{1}$$

where the input $u(t)$ is a sine wave with an amplitude A:

$$u(t) = A\sin(\omega_0 t) \quad \rightarrow \quad U(s) = A\frac{\omega_0}{s^2+\omega_0^2} \tag{2}$$

With zero initial conditions, we have

$$Y(s) = G(s)A\frac{\omega_0}{s^2+\omega_0^2} \tag{3}$$

Frequency response

$$Y(s) = G(s)A\frac{\omega_0}{s^2 + \omega_0^2} \tag{4}$$

Using partial-fraction expansion, the solution takes the form of

$$Y(s) = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \ldots + \frac{\alpha_n}{s - p_n} + \frac{\alpha_0}{s + j\omega_0} + \frac{\alpha_0^*}{s - j\omega_0} \tag{5}$$

where α_0^* is the complex conjugate of α_0.

The time response is

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \ldots + \alpha_n e^{p_n t} + 2|\alpha_0| \sin(\omega_0 t + \phi) \tag{6}$$

with

$$\phi = \tan^{-1}\left(\frac{\Im(\alpha_0)}{\Re(\alpha_0)}\right) \tag{7}$$

Frequency response

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \ldots + \alpha_n e^{p_n t} + 2|\alpha_0| \sin(\omega_0 t + \phi) \tag{8}$$

Provided that $p_i < 0 \ \forall \ i$, the exponential terms die out eventually and the steady-state response $y(t \to \infty)$ is

$$y(t) = 2|\alpha_0| \sin(\omega_0 t + \phi) \tag{9}$$

which can be expressed as

input magnitude.

$$y(t) = AM \sin(\omega_0 t + \phi) \tag{10}$$

where

magnitude

$$\boxed{M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0} = \sqrt{[\Re G(j\omega_0)]^2 + [\Im G(j\omega_0)]^2}}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{\Im(G(j\omega_0))}{\Re(G(j\omega_0))}\right) = \angle G(j\omega_0)}$$

phase

Frequency response

$$G(j\omega_0) = R(\omega_0) + jX(\omega_0)$$

steady-state only

$$M = |G(s)| = \sqrt{R(\omega_0)^2 + X(\omega_0)^2}$$

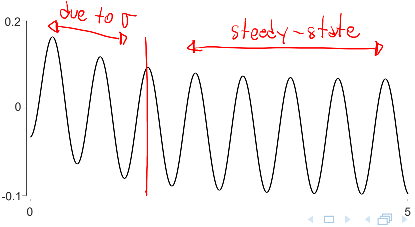
$$\angle G(j\omega_0) = \tan^{-1}\left(\frac{X(\omega_0)}{R(\omega_0)}\right)$$

Note that in the above, $\boxed{s = j\omega}$ rather than $s = \sigma + j\omega$. Why?

Back to the temporal response, what can we conclude?

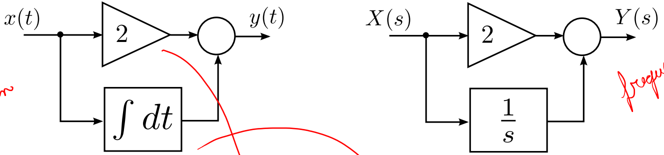
$$y(t) = AM\sin(\omega_0 t + \phi)$$

frequency response
$s = j\omega$
rather than $\sigma + j\omega$



Example

Consider $x(t) = \sin(0.5t)$. What is $y(t)$?



time domain

frequency domain

$$y(t) = 2\sin(0.5t) + \int \sin(0.5t)dt$$

$$y(t) = 2\sin(0.5t) - 2\cos(0.5t)$$

Trigonometric identity: $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \tan^{-1} b/a)$

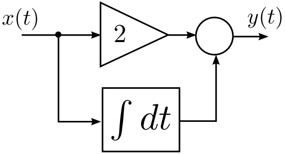
gain $= \frac{2\sqrt{2}}{A=1} = 2\sqrt{2}$

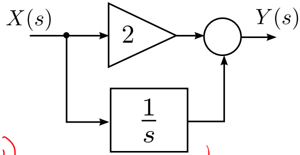
$$y(t) = 2\sqrt{2}\sin(0.5t - 0.785)$$

phase

Example

What is the magnitude of $Y(s)$?





$$Y(s) = \overbrace{\left(2 + \frac{1}{s}\right)}^{G(j\omega)} X(s)$$

$\omega = 0.5$ rad/s.

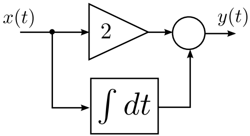
$$|G(s)| = \left(2 + \frac{1}{0.5j}\right) = \left(2 + \frac{1}{0.5j} \times \frac{j}{j}\right)$$

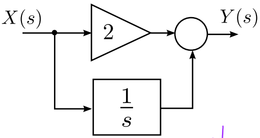
$$|G(s)| = |2 - 2j|$$

$$|G(s)| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$ gain

$$\angle G(s) = \tan^{-1}(-2/2) = 0.785 \text{ (rad)}$$ phase shift

Frequency response





real

$$G(s) = 2 + \frac{1}{s} \rightarrow \boxed{G(j\omega) = 2 - \frac{1}{\omega}j}$$

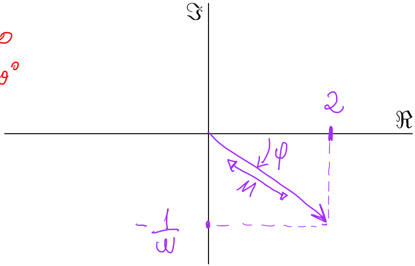
Imaginary

$Re = 2 \;\; \forall \omega$

$Im = -\frac{1}{\omega}$

if $\omega \rightarrow 0 \rightarrow \begin{cases} M \rightarrow \infty \\ \varphi \rightarrow -90^\circ \end{cases}$

if $\omega \rightarrow \infty \rightarrow \begin{cases} M \rightarrow 2 \\ \varphi \rightarrow 0 \end{cases}$

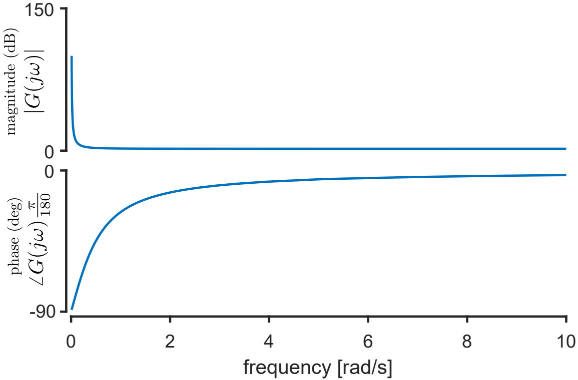


Magnitude and phase plots

Recall that $G(j\omega) = 2 + \frac{1}{\omega}j$.

We can plot the magnitude and gain for any given frequency range.

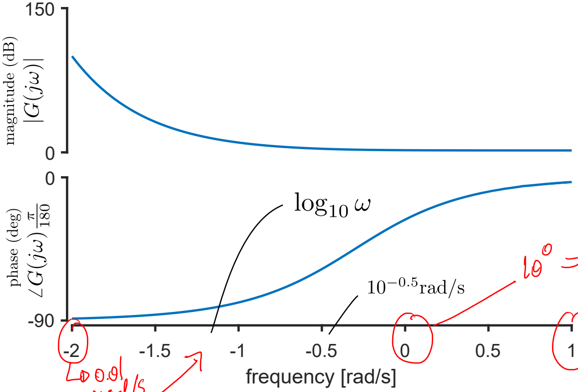
same



Magnitude and phase plots

The horizontal axis is typically logarithmic

The horizontal axis shows $\log_{10}(\omega)$



$\log_{10}(|G(j\omega)|)$

$\to 0.01$ rad/s

$10^0 = 1$ rad/s

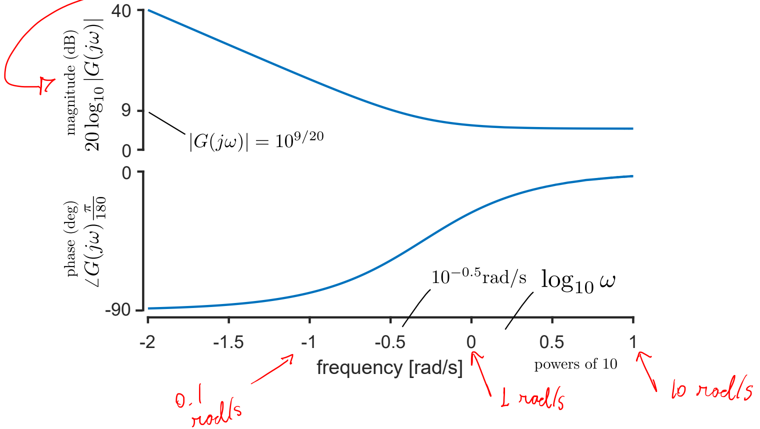
$\to 10^1 = 10$ rad/s

powers of 10

Bode plot

The magnitude is expressed in Decibels

The magnitude is shown as $20\log_{10}|G(j\omega)|$. This is called the "gain".



Gain and phase

Given a transfer function

$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{11}$$

The gain is

$$|G(j\omega)| = 20\log\left[k\frac{\prod_{i=1}^{n}(j\omega+z_i)}{\prod_{i=1}^{m}(j\omega+p_i)}\right] \tag{12}$$

Since $\boxed{\log(a\times b) = \log(a) + \log(b)}$, we can rewrite the gain as

$$[G(j\omega)| = 20\log(k) + \sum_{i=1}^{n}\left[20\log(j\omega+z_i)\right] + \sum_{i=1}^{m}\left[20\log\frac{1}{(j\omega+p_i)}\right] \tag{13}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.

Gain and phase

Given a transfer function

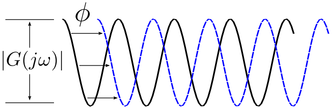
$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{14}$$

The phase is

$$\angle[G(j\omega)] = \phi = \tan^{-1}\left[\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]}\right] \tag{15}$$

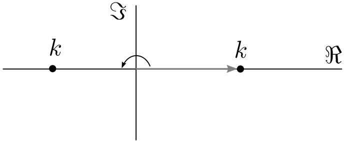
$$\phi = \angle(k) + \sum_{i=1}^{n}[\angle(j\omega+z_i)] + \sum_{i=1}^{m}\left[\angle\frac{1}{j\omega+p_i}\right] \tag{16}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.



Bode plot building blocks

Constant gain $G(s) = k$



$$\boxed{|G(\omega)| = \ 20 \log |k| \quad \forall \ \omega}$$

$$\boxed{\phi(\omega) = \tan^{-1}\left(\frac{0}{k}\right) = 0 \quad \forall \ \omega}$$

gain and phase are frequency-independent

The gain curve is a horizontal line on the Bode plot.

If $k < 0$, the magnitude remains $20 \log |k|$ and the phase becomes -180°.

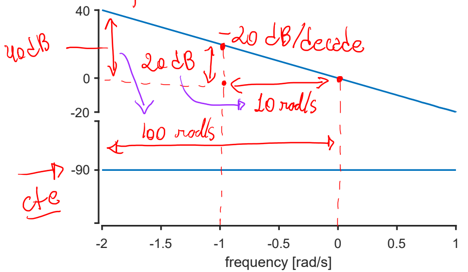
Bode plot building blocks

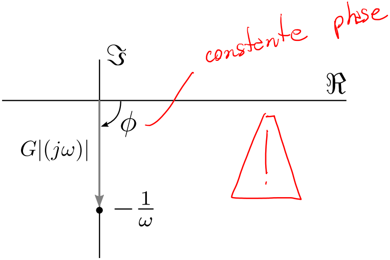
Pole at the origin $G(s) = \frac{1}{s}$

when frequency increases by a factor of 10, the gain decreases by 20 dB

$$|G(\omega)| = 20\log\left|\frac{1}{j\omega}\right| = 20\log(1) - 20\log\omega = \boxed{-20\log\omega}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{-1/\omega}{c}\right)_{c\to 0} = -90^\circ$$



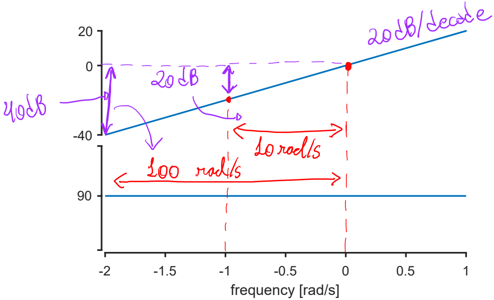


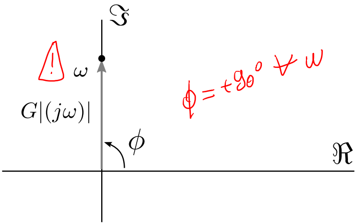
Bode plot building blocks

Zero at the origin $G(s) = s$

$$\boxed{|G(\omega)| = \ 20\log|j\omega| = 20\log\omega}$$

$$\boxed{\phi(\omega) = \tan^{-1}\left(\frac{\omega}{c}\right)_{c\to 0} = 90^\circ}$$



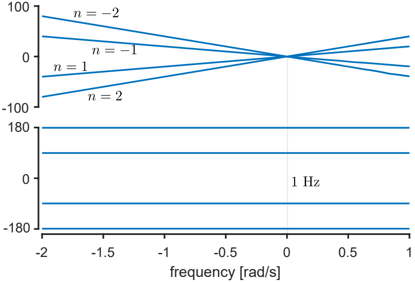


Bode plot building blocks

Multiple zeros or poles at the origin $G(s) = s^n$, $n \geq 0$ or $n \leq 0$

$$\boxed{|G(\omega)| = \; 20\log|(j\omega)^n| = n \times 20\log\omega}$$

$$\boxed{\phi(\omega) = \tan^{-1}\left(\frac{\omega}{c}\right)_{c\to 0} = n \times 90^\circ}$$



Exercise 93

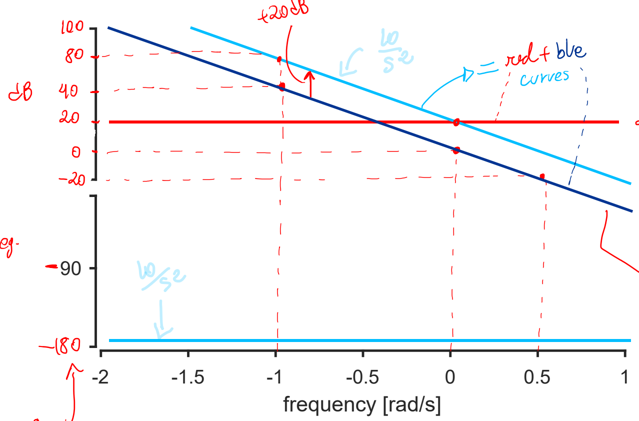
$20\log\left(\frac{10}{s^2}\right) = 20\log(10) + 20\log\left(\frac{1}{s}\right) + 20\log\left(\frac{1}{s}\right)$

$s = j\omega$

$= 20\log(10) + 40\log\left(\frac{1}{s}\right)$

40 dB/decade

Sketch the Bode diagram for the function $G(s) = \frac{10}{s^2}$



20 log(10) = 20 dB

when $\omega = 1$

$20\log\left|\frac{1}{\omega}\right| = 0$

40 dB/dec

2 poles at the origin

-90 -90

2 poles at the origin

Bode plot building blocks

Poles on the real axis $G(s) = \frac{1}{\frac{s}{\omega_0}+1}$, $G(j\omega) = \frac{1}{j\frac{\omega}{\omega_0}+1}$

The magnitude of $G(s)$ is

$$|G(j\omega)| = \left|\frac{1}{j\frac{\omega}{\omega_0}+1}\right| = \left|\frac{1}{j\frac{\omega}{\omega_0}+1} \times \frac{-j\frac{\omega}{\omega_0}+1}{-j\frac{\omega}{\omega_0}+1}\right| \tag{17}$$

$$|G(j\omega)| = \left|\underbrace{\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2+1}}_{\text{real}} - j\underbrace{\frac{\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2+1}}_{\text{imaginary}}\right| = \sqrt{\left(\underbrace{\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2+1}}_{\text{real}}\right)^2 + \left(\underbrace{\frac{-\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2+1}}_{\text{imaginary}}\right)^2} \tag{18}$$

$$|G(j\omega)| = \sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^2+1}{\left[\left(\frac{\omega}{\omega_0}\right)^2+1\right]^2}} = \sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2}+1}}$$

Bode plot building blocks

Poles on the real axis $G(s) = \frac{1}{\frac{s}{\omega_0}+1}$, $G(j\omega) = \frac{1}{j\frac{\omega}{\omega_0}+1}$

$$|G(j\omega)| = \sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2}+1}}$$

In Decibels, the gain is

magnitude

$$|G(j\omega)|_{dB} = 20\log\sqrt{\frac{1}{\frac{\omega^2}{\omega_0^2}+1}} = 20\log\left(\frac{\omega^2}{\omega_0^2}+1\right)^{-\frac{1}{2}} = \boxed{-20\log\sqrt{\frac{\omega^2}{\omega_0^2}+1}}$$

From Equation (18), the phase $\phi = \tan^{-1}\Im/\Re$ is

phase

$$\phi = \tan^{-1}\left[-\frac{\frac{\frac{\omega}{\omega_0}}{\left(\frac{\omega}{\omega_0}\right)^2+1}}{\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2+1}}\right] = \boxed{\tan^{-1}\left(-\frac{\omega}{\omega_0}\right)}$$

Bode plot building blocks

$$|G(j\omega)|_{dB} = -20\log\sqrt{\frac{\omega^2}{\omega_0^2}+1} \qquad \phi = \tan^{-1}\left(-\frac{\omega}{\omega_0}\right)$$

Case 1: $\omega << \omega_0$. Thus $\omega^2/\omega_n^2 + 1 \approx 1$

$\rightarrow$ The gain is $-20\log(1) = 0$ dB

$\rightarrow$ The phase is $\tan^{-1}(0) = 0^\circ$

Case 2: $\omega = \omega_0$

$\rightarrow$ The gain is $-20\log(\sqrt{2}) = -3$ dB

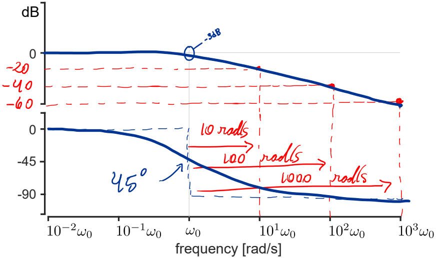
$\rightarrow$ The phase is $\tan^{-1}(-1) = -45^\circ$

Case 3: $\omega >> \omega_0$, $\therefore (\omega/\omega_0)^2 >> 1$

$\rightarrow$ The gain is $-20\log\left(\sqrt{(\omega/\omega_0)^2}\right)$ dB

$\rightarrow$ The phase is $\tan^{-1}(\infty) = -90^\circ$

-20dB/decade



Exercise 94

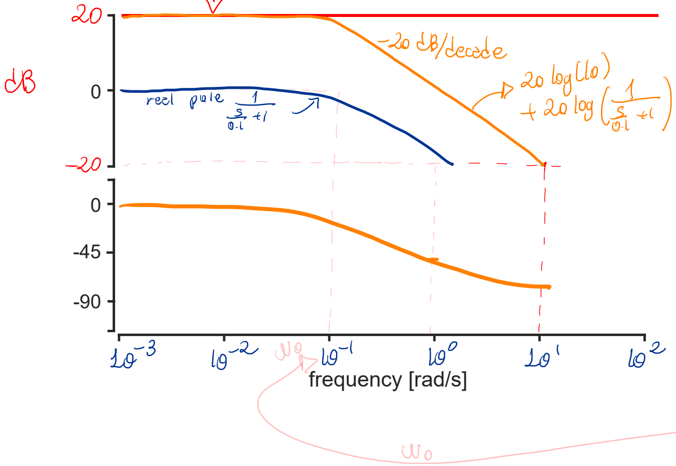
$20\log(10)$

cte gain + real pole

Sketch the Bode diagram for the function $G(s) = \frac{1}{s+0.1}$ $= \frac{1}{0.1(s+1)} = \frac{10}{\frac{s}{0.1}+1}$

$\omega_0 = 0.1$ rad/s

→ cut off frequency



Bode plot building blocks

Zeros on the real axis $G(s) = \frac{s}{\omega_0} + 1,\ G(j\omega) = 1 + j\frac{\omega}{\omega_0}$
The gain is

$$|G(j\omega)| = 20\log\sqrt{\left(\frac{\omega}{\omega_0}\right)^2 + 1} \tag{19}$$

Note that the above is equal the negative gain of a pole on the real axis.

The phase is

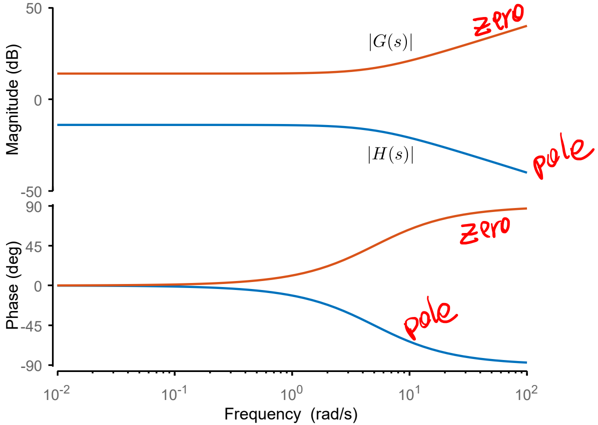
$$\phi = \tan^{-1}\left(\frac{\omega}{\omega_0}\right) \tag{20}$$

Recall that the phase for a real pole was $\tan^{-1}\left(-\frac{\omega}{\omega_0}\right)$

The real zero is the **negative of a real pole** on the Bode plot

Bode plot building blocks

$H(s) = \frac{1}{s+5}$, $G(s) = s + 5$,



Exercise 95

A tendon-operated robotic hand can be implemented using a pneumatic actuator. The actuator can be represented by

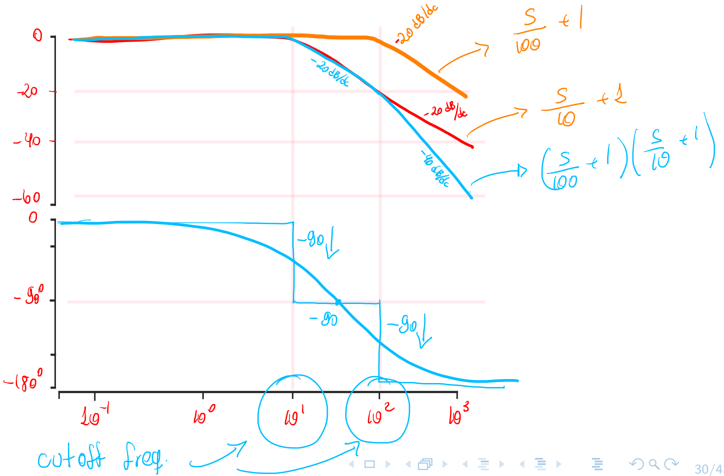
$$G(s) = \frac{1000}{(s+100)(s+10)} \tag{21}$$

Plot the frequency response of $G(j\omega)$. Calculate the magnitude in dB of $G(j\omega)$ at $\omega = 10$ rad/s $\omega = 200$ rad/s.

Exercise 95 - continued

$$G(s) = \frac{1}{\left(\frac{s}{100}+1\right)\left(\frac{s}{10}+1\right)}$$ (see next slide)

$$G(s) = \frac{1000}{(s+100)(s+10)} \tag{22}$$



Exercise 95 - continued

$$G(s) = \frac{1000}{(s+100)(s+10)} \tag{23}$$

Magnitude at 10 rad/s

$$G(j\omega) = \frac{1000}{(10j+100)(10j+10)}$$

$$20\log(|G(j\omega)|) =$$

$$20\log(1000) +$$
$$-20\log\sqrt{10^2+100^2} +$$
$$-20\log\sqrt{10^2+10^2}$$

$$\boxed{|G(j\omega)|_{dB} = -3.05\,dB}$$

$$G(s) = \frac{1000}{100\left(\frac{s}{100}+1\right)10\left(\frac{s}{10}+1\right)}$$

$$G(s) = \frac{\cancel{1000}}{\cancel{1000}\left(\frac{s}{100}+1\right)\left(\frac{s}{10}+1\right)}$$

cutoff frequencies $\begin{cases} 100 \text{ rad/s} \\ 10 \text{ rad/s} \end{cases}$

Exercise 96

Plot the Bode magnitude and phase for the system with the transfer function

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

Procedure

Identify the components of the transfer function for which you know the Bode plot. Draw the Bode plot for each component, then add them up to find the Bode plot of $G(s)$.

Exercise 96 - continued

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)} = \frac{2\left(\frac{s}{0.5}+1\right)}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$$ (see next slide)

→ Constant: $20\log(2) \cong 6\text{dB}$

→ Pole at the origin: $s \to -20\ \text{dB/dec.}$

→ Real zero: $\frac{s}{10}+1 \to -20\text{dB/dec}\quad \omega > 10$, $\frac{s}{50}+1 \to -20\text{dB/dc}\quad \omega > 50$

→ Real pole: $\frac{s}{0.5}+1 \to +20\text{dB/dec}\quad \omega > 0.5$

The gain is

$$G = 20\log\left[\frac{2000(j\omega+0.5)}{j\omega(j\omega+10)(j\omega+50)}\right]$$

Which can be expressed as

$$20\log(2000) + 20\log\sqrt{\omega^2+0.5^2} - 20\log\omega - 20\log\sqrt{\omega^2+10^2} - 20\log\sqrt{\omega^2+50^2}$$

Exercise 96 - continued

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)} = \frac{2000 \cdot 0.5\left(\frac{s}{0.5}+1\right)}{s\,10\left(\frac{s}{10}+1\right)50\left(\frac{s}{50}+1\right)}$$

$$G(s) = \frac{2\left(\frac{s}{0.5}+1\right)}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$$

cut off frequencies $\rightarrow \begin{cases} 0.5 \\ 10 \\ 50 \end{cases}$ rad/s

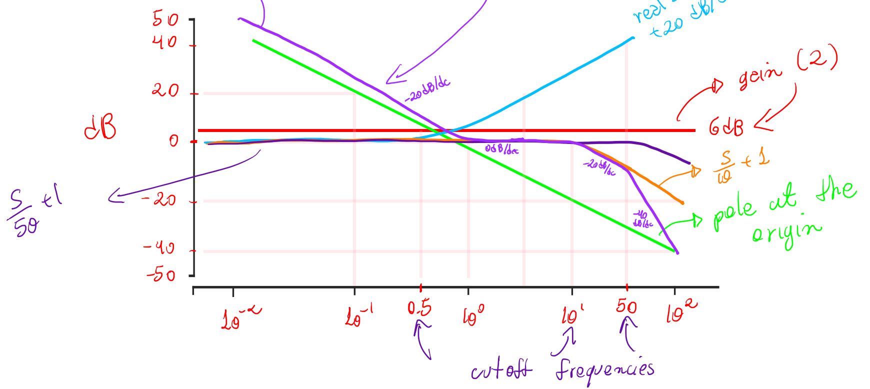
Exercise 96 - continued

$20\log(2) = 6\,dB$

sum of all other lines.

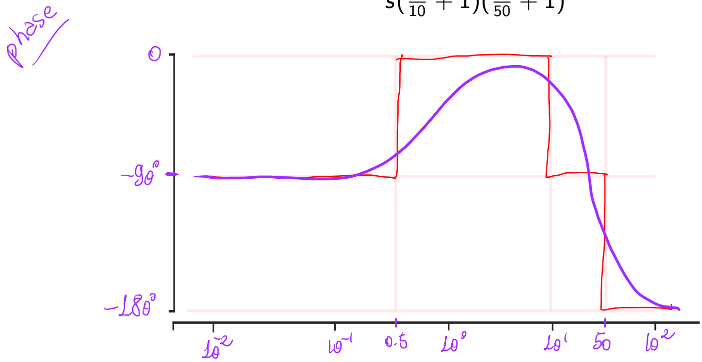
$$G(s) = \frac{2(\frac{s}{0.5}+1)}{s(\frac{s}{10}+1)(\frac{s}{50}+1)}$$

real zero +20 dB/dec after 0.5 rad/s



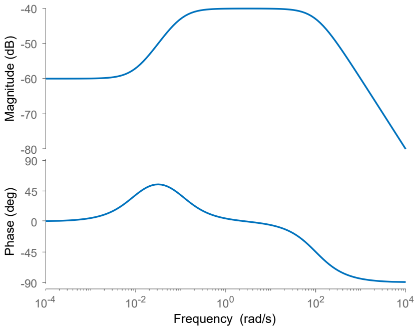
Exercise 96 - continued

$$G(s) = \frac{2(\frac{s}{0.5}+1)}{s(\frac{s}{10}+1)(\frac{s}{50}+1)}$$



Exercise 97

The body plot of a system was determined experimentally using a sinusoidal wave excitation. Estimate its transfer function.

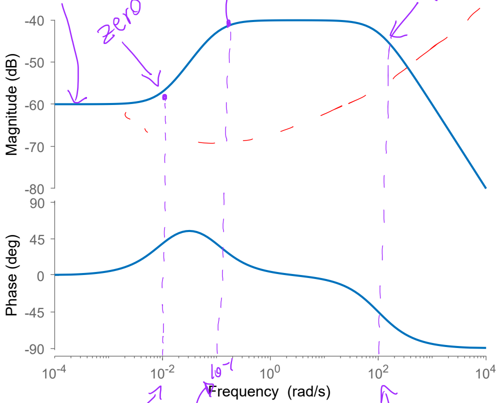


Exercise 97 - continued

$20\log(K)=-60$

$K=10^{\frac{-60}{20}}$

$K=0.001$



$\frac{s}{0.01}+1$ $\qquad$ $\frac{1}{\frac{s}{0.1}+1}$ $\qquad$ $\frac{1}{\frac{s}{100}+1}$

0.001

$$G(s)=\frac{K\left(\frac{s}{0.01}+1\right)}{\left(\frac{s}{0.1}+1\right)\left(\frac{s}{100}+1\right)}$$

$$G(s)=\frac{0.001\left(\frac{1}{0.01}\right)(s+0.01)}{\frac{1}{0.1}(s+0.1)\frac{1}{100}(s+100)}$$

$$\boxed{G(s)=\frac{(s+0.01)}{(s+0.1)(s+100)}}$$

Exercise 98

A robotic arm has a joint-control loop transfer function

home work
check solution
with matlab

$$G(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)}.$$

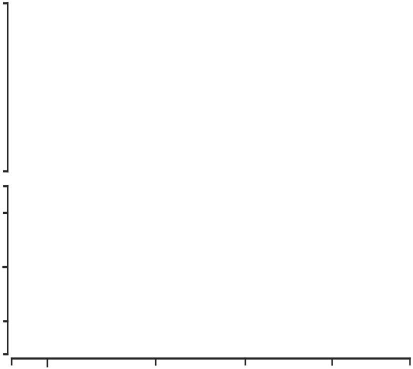
$\boxed{\text{bode}\ (G)}$

Plot the Bode diagram of the transfer function.

Exercise 98 - continued

Homework

$$G(s) = \frac{300(s+100)}{s(s+10)(s+40)}$$



Exercise 99

A typical industrial robot has multiple degrees of freedom. A unity feedback position control system for a force-sensing joint has a loop transfer function

Homework

$$W(s) = \frac{10}{\left(\frac{s}{4}+1\right)\left(\frac{s}{20}+1\right)(s+1)\left(\frac{s}{80}+1\right)}.$$

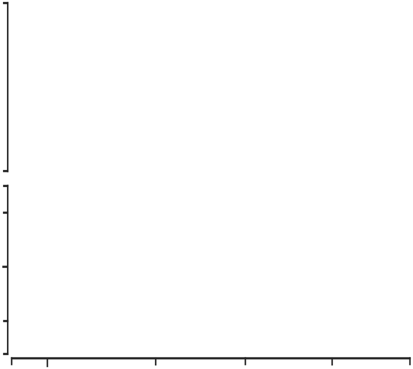
Sketch the Bode diagram of this system.

Check solution
with MATLAB
bode(W)

Exercise 99 - continued

$$W(s) = \frac{10}{(\frac{s}{4}+1)(\frac{s}{20}+1)(s+1)(\frac{s}{80}+1)}.$$

Home work



Next class...

- More on Bode plots