MECE 3350U
Control Systems

Lecture 12

The Root-Locus Method 2/2

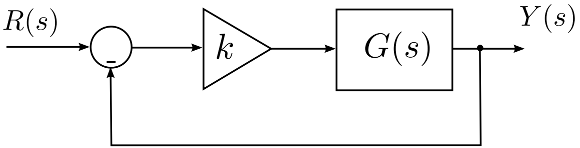
Outline of Lecture 12

By the end of today's lecture you should be able to

- Extend the concept of root locus
- Understand the applications of the Root-Locus method
- Apply Root-Locus method to a design problem

Review

Consider the following closed-loop system:



The closed-loop transfer function is

$$T(s) = \frac{kG(s)}{1 + kG(s)} \tag{1}$$

And the characteristic equations is

$$1 + kG(s) = 0 \tag{2}$$

Review

To analyse the influence of a given parameter of interest k, the characteristic equation of the closed-loop system must in the format

$$1 + kH(s) = 0 \tag{3}$$

$\Rightarrow$ k is the parameter of interest

$\Rightarrow$ $H(s)$ is a function of s

$$1 + k\frac{P(s)}{Q(s)} = 0 \tag{4}$$

$Q(s) + kP(s) = 0$

$\frac{1}{k}Q(s) + P(s) = 0$

If $k = 0$, the poles of $T(s)$ are the roots of $Q(s)$

If $k \to \infty$, the poles of $T(s)$ are the roots of $P(s)$

The root locus is the set of values of s for which $1 + kH(s) = 0$ is satisfied as the real parameter k varies from 0 to ∞.

Angle requirement

$$1 + kG(s) = 0, \;\; kG(s) = -1 + j0 \tag{5}$$

If the open loop transfer function is

$$G(s) = k\frac{(s+z_1)(s+z_2)(s+z_3)\dots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)} \tag{6}$$

The magnitude requirement for root locus is

$$|G(s)| = k\frac{|s+z_1||s+z_2||s+z_3|\dots|s+z_m|}{|s+p_1||s+p_2||s+p_3|\dots|s+p_n|} = 1 \tag{7}$$

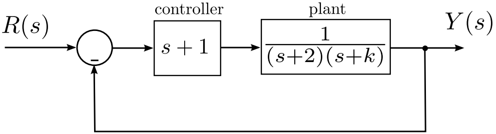
The angle requirement for root locus is

$$\begin{aligned} \angle G(s) = & \quad \angle(s+z_1) + \angle(s+z_2) + \dots \\ & - [\, \angle(s+p_1) + \angle(s+p_2) + \dots \,] = 180^\circ + \ell 360^\circ \end{aligned}$$

where $\ell = 1, 2, 3$ ...

Example 1

Sketch the root locus of the closed loop system as k varies from 0 to ∞.



Find the closed-loop transfer function.

$$T(s) = \frac{\frac{(s+1)}{(s+2)(s+k)}}{1 + \frac{(s+1)}{(s+2)(s+k)}}$$

Prepare the characteristic equation as $1 + kH(s)$.

$$1 + \frac{s+1}{(s+2)(s+k)} = 0 \quad \rightarrow \quad (s+2)(s+k) + s + 1 = 0$$

$$s^2 + 3s + 1 + k(s+2) = 0$$

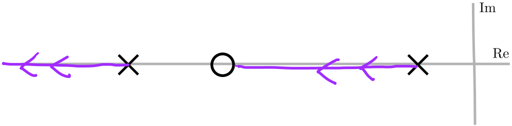
$$1 + \frac{k(s+2)}{s^2 + 3s + 1} = 0$$

Example 1 - continued

$$1 + k\frac{s+2}{s^2+3s+1} = 1 + k\frac{s+2}{(s+2.618)(s+0.382)}$$

Locate the poles and zeros of $H(s)$ as defined in (2)

Draw the approximate root locus



What can we conclude?

Example 2

Find the root locus for as system whose characteristic equation is

$$1 + k\frac{s+1}{s^3+s^2+3s+1}.$$

The above can be rewritten as

$$1 + k\frac{s+1}{(s+0.36)(s+0.32+j1.63)(s+0.32-j1.63)}.$$

$n - m = 2$

$q = 1$

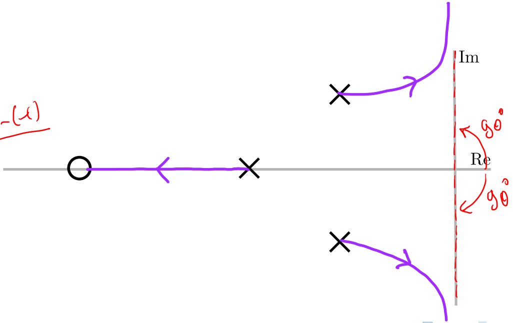
$\theta_1 = 90^\circ$

$q = 2$

$\theta_2 = -90^\circ$

$$\alpha = \frac{-0.36 - 0.32 - 0.32 - (-1)}{2}$$

$\alpha = 0$



Note: Rule #5 only applies to real poles and zeros

Breakaway point

For the characteristic equation

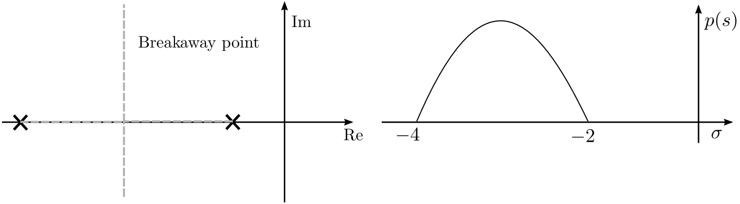
$$1 + k\frac{Q(s)}{P(s)} = 0 \tag{8}$$

Let $k = p(s)$ and rearrange (8) so that

$$p(s) = -\frac{P(s)}{Q(s)} \tag{9}$$

The breakaway point satisfies

$$\frac{dp(s)}{ds} = \frac{dk}{ds} = 0 \tag{10}$$



Breakaway point - Example

Find the breakaway point for

$$1 + k\frac{1}{(s+2)(s+4)} = 0. \qquad (11)$$

Let $k = p(s)$ and rewrite (11) as in (9)

$$p(s) = -(s+2)(s+4) = -(s^2+6s+8) \qquad (12)$$

$1 + p(s) = \frac{1}{(s+2)(s+4)}$

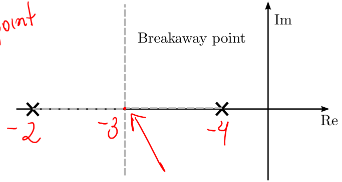
$p(s) = -(s+2)(s+4)$

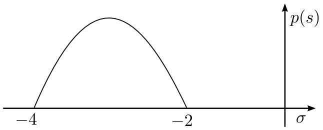
$\frac{dp(s)}{ds} = -(2s+6) = 0$

$s = -3$ ← break point

Note:

$p(-3)$ gives K at the breakaway point





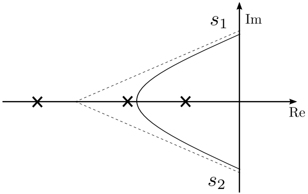
Imaginary axis crossing point

The point where the root locus crosses the imaginary axis indicates the maximum gain k_{max} before instability.

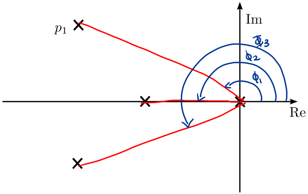
How to find k_{max}? $\Rightarrow$ Use the Routh-Hurwitz method. Example:

$$1 + k\frac{1}{(s+1)(s+2)(s+3)} = 0 \rightarrow s^3 + 6s^2 + 11s + 6 + k = 0$$

s_3	1	11
s_2	6	$6+k$
s_1	$\frac{6+k-66}{-6}$	0
s_0	$6+k$	0



Departure angle from a pole

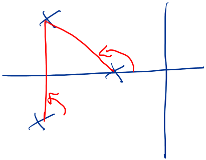


In what direction does p_1 move?

Recall the angle requirement:

$$\angle G(s) = (s + z_1) + \angle(s + z_2) + \dots - [\, \angle(s + p_1) + \angle(s + p_2) + \dots \,] = 180^\circ + \ell 360^\circ$$

$$\sum \psi - \sum \phi = 180^\circ + 360^\circ(\ell)$$



where
→ ψ is the angle from a point to a zero;

→ ϕ is the angle from a point to a pole;

For any point in the plane we can write

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

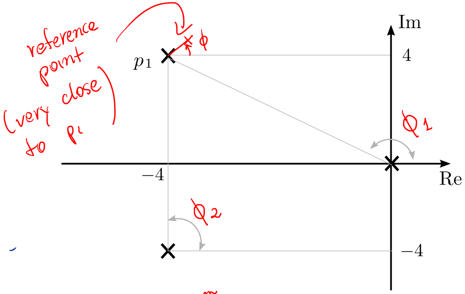
ℓ is an integer.

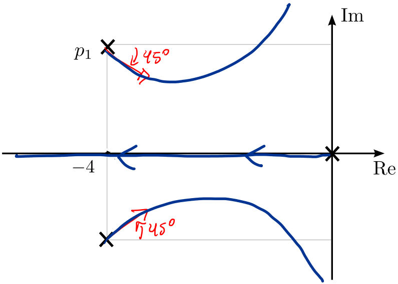
Departure from a pole

In what direction does p_1 move?

$\phi_1 = 180 - \text{atan}(4/4)$

$\phi_1 = 135^\circ$, $\phi_2 = 90^\circ$





angle between p_1 and the reference point

zeros poles

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

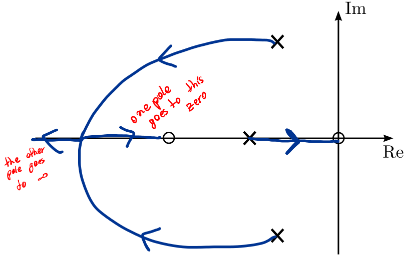
(ℓ is an integer)

$\phi = 0 - (135^\circ + 90^\circ) - 180^\circ - (-1)(360^\circ)$

$\phi = -45^\circ$

Arrival at a zero

The same approach can be used to determine the angles of arrival of a branch at a zero.



$$q\psi = \sum \phi - \sum \psi + 180^\circ + \ell 360^\circ$$

The 10 rules for drawing the root-locus

Rule 1: As k varies from 0 to ∞, there are n lines (loci) where n is the degree of $Q(s)$ or $P(s)$, whichever is greater.

Rule 2: As k varies from 0 to ∞, the roots of the characteristic equation move from the poles to the zeros of $H(s)$.

Rule 3: The root loci must be symmetrical with respect to the horizontal axis.

Rule 4: The a root cannot cross its path

Rule 5: The loci are on the real axis to the left of an odd number of poles and zeros.(that lie on the real axis)

Rule 6: Lines leave (break out) and enter (break in) the real axis at 90°

The 10 rules for drawing the root-locus

Rule 7: If there a different number of poles and zeros, extra lines that do not have a pair go to or come from infinity.

Rule 8: The angle of the asymptotes of the curves that go to infinity is

$$\phi = \frac{180^\circ + 360^\circ(q-1)}{n-m}, \;\; q = 1, 2, \ldots, n-m \tag{13}$$

and they radiate out from the real axis at

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \tag{14}$$

Rule 9: If there is a least two lines that go to infinity, then the sum of all the roots is constant.

Rule 10: If the gain sweeps from 0 to $-\infty$, the root loci can be drawn by reversing Rule 5 **and** adding a 180° to the asymptote angles.

Steps for drawing the root locus

Step 1

Prepare the characteristic equation in the form of

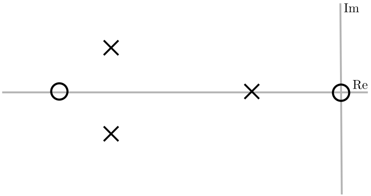
$$1 + kH(s) = 0 \tag{15}$$

and factor the m poles and n zeros

$$1 + k\frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} \tag{16}$$

Step 2

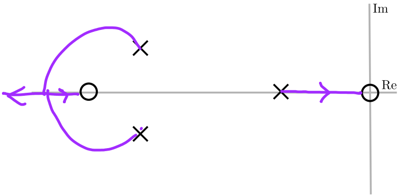
Locate the poles and zeros of $H(s)$ in the plane



Steps for drawing the root locus

Step 3

Locate the segments of the of the real axis that are root loci. Root loci are to the left of an odd number of poles and zeros.



Step 4

Calculate the angle θ and centre α of asymptotes of loci that tend to infinity

$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

Steps for drawing the root locus

Step 5

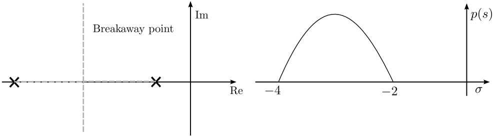
If applicable, determine the points at which the loci cross the imaginary axis. Use Routh-Hurwitz criterion.

s_3	a_3	a_1
s_2	a_2	a_0
s_1	b_1	0
s_0	c_1	0

Step 6

If applicable, determine the breakaway point on the real axis. Set $p(s) = k$, and find the minimum and/or maximum values of $p(s)$, i.e,:

$$p(s) = -\frac{1}{H(s)}, \rightarrow \frac{d}{ds}\left[-\frac{1}{H(s)}\right] = 0$$



Steps for drawing the root locus

Step 7

Determine the angle of locus departure from complex poles and the angle of locus at arrival at complex zeros using the phase criterion.

Departure from a complex pole:

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

Arrival at a complex zero

$$q\psi = \sum \phi - \sum \psi + 180^\circ + \ell 360^\circ$$

Step 8

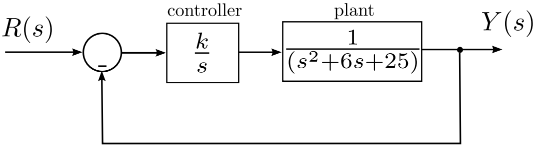
Complete the root locus

Step 9

You may check you results using the Matlab function "rlocus(H);".

Exercise 60

Determine the root-locus plot for the following closed-loop system.



Procedure:

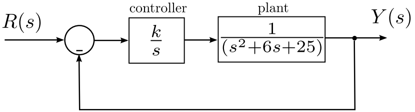
Follow the steps given in this lecture.

Use the provided Matlab code to check your answer (posted on B.B.)

Exercise 60 - continued

$$1 + K\frac{1}{s(s^2+6s+25)}$$

Step 1: Preparing the characteristic equation.

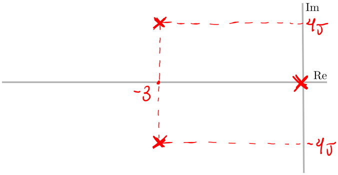


Step 2: Poles and zeros

$s = 0$

$s = -3 \pm 4j$

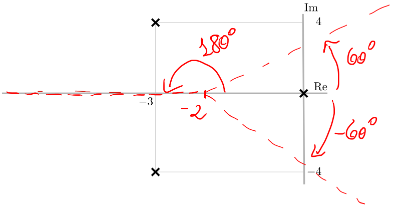
Step 3: Locate the segments of the loci



Exercise 60 - continued

Step 4: The angle and centre of asymptotes

$n - m = 3$



$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

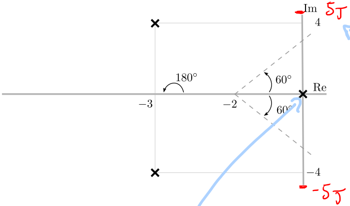
$q = 1$, $\theta_1 = 60^\circ$ $\quad$ $q = 2$, $\theta_2 = 180^\circ$ $\quad$ $q = 3$, $\theta_3 = -60^\circ$

$\alpha = \frac{-3-3+0}{3} = -2$

Exercise 60 - continued

$s^3 + 6s^2 + 25s + K = 0$ ← root-locus path.

replace $s = j\omega$



Step 5: Points where the loci cross the imaginary axis

$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = 0$

$-j\omega^3 - 6\omega^2 + 25j\omega + K = 0$

$\underbrace{-6\omega^2 + K}_{Re} + \underbrace{j\omega(25 - \omega^2)}_{Im} = 0$

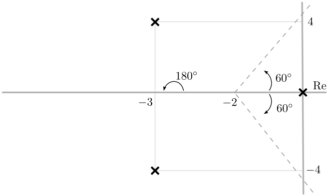
↳ Set $Re = 0$

thus $\omega(25 - \omega^2) = 0$

$\boxed{\omega = 0}$

$\boxed{\omega = \pm 5}$

Exercise 60 - continued



Step 6: Breakaway points

$$G(s) = \frac{1}{s(s^2 + 6s + 25)} \tag{17}$$

$$p(s) = -\frac{1}{G(s)}$$

$$p(s) = -s(s^2+6s+25)$$

$$\frac{dp(s)}{ds} = -(3s^2+12s+25) = 0$$

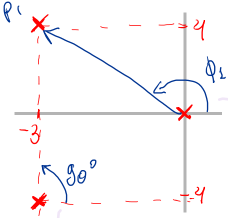
$$s = -2 \pm 2.08j$$

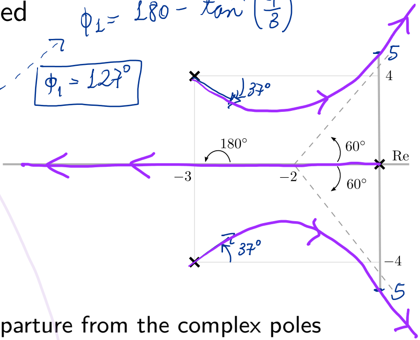
Since the result is an imaginary number, there is NO breakaway point.

Exercise 60 - continued

$\phi_1 = 180 - \tan^{-1}\left(\frac{4}{3}\right)$

$\boxed{\phi_1 = 127^\circ}$





Step 7: Angle of departure from the complex poles

$\phi = \sum \psi - \sum \phi - 180^\circ - l(360^\circ)$

l is an integer

$l = 1$

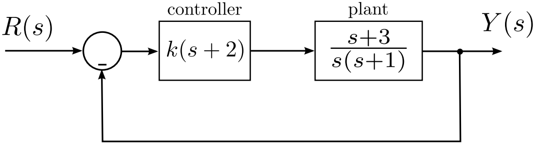
$\phi = 0 - (90^\circ + 127^\circ) - (180 + 360^\circ)$

$\boxed{\phi = -37^\circ}$

→ departure of p1

Exercise 61

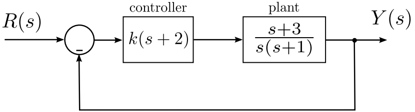
Sketch the root loci for the system shown. Determine the range of k for which the closed-loop system is underdamped.



Follow the steps given in this lecture.

Determine the range of k for which the roots are complex.

Exercise 61 - continued



$$1 + K\frac{(s+2)(s+3)}{s(s+1)} = 0$$

$$K = \frac{-s(s+1)}{(s+2)(s+3)}$$

break away point

$$p(s) = K$$

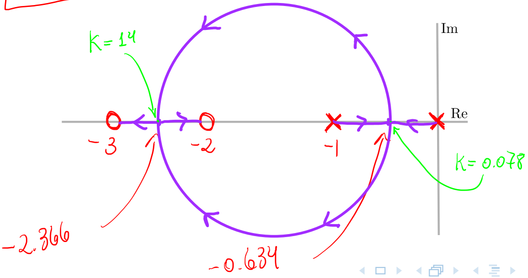
$$p(s) = \frac{-s(s+1)}{(s+2)(s+3)}$$

$$\frac{dp(s)}{ds} = 0 \Longrightarrow \begin{cases} s = -0.634 \\ s = -2.366 \end{cases}$$

when

$s = -0.634$
$p(s) = K = 0.078$

$s = -2.366$
$p(s) = K = 14$



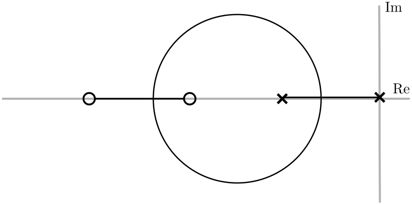
The system is underdamped for

$0.078 < K < 14$

critically damped for $K = 14$, $K = 0.0718$

overdamped if $K > 14$ or $K < 0.0718$

Exercise 61 - continued

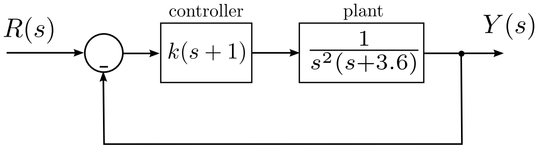


$$k = -\frac{s(s+1)}{(s+3)(s+2)}$$

See previous slide.

Exercise 62

Sketch the root loci for the system shown.



Follow the steps given in this lecture.

Exercise 62 - continued

$n - m = 2$

$q = 1$ $\quad$ $q = 2$

$\theta_1 = 90^\circ$ $\quad$ $\theta_2 = -90^\circ$

$$H(s) = \frac{s+1}{s^2(s+3.6)}$$

$$\alpha = \frac{\sum p - \sum z}{n-m} = \frac{0 + 0 - 3.6 - (-1)}{2}$$

$$\boxed{\alpha = -1.3}$$

$$1 + K \frac{(s+1)}{s^2(s+3.6)} = 0$$

$K = p(s)$

$$p(s) = \frac{-s^3 - 3.6s^2}{s+1}$$

$$\frac{dp(s)}{ds} = 0 \rightarrow s = 0 \quad \text{(break away point)}$$

$$\rightarrow s = -1.65 \pm 0.9367j$$

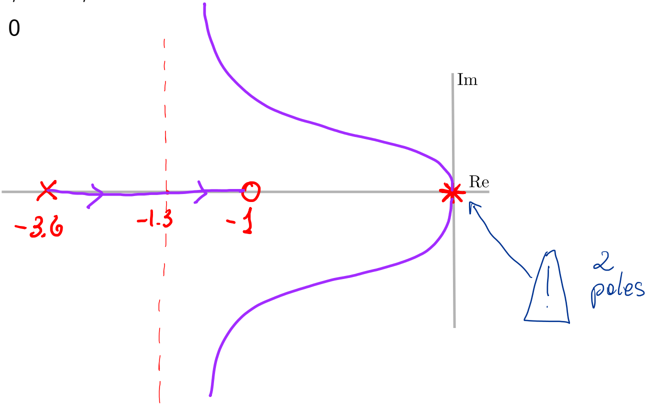
Not a break away point (imaginary)

Exercise 62 - continued

$$H(s) = \frac{s+1}{s^2(s+3.6)}$$

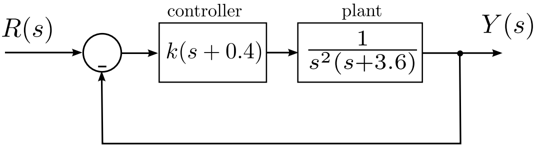
Asymptotes: 90°, -90°, at $s = -1.3$.

Breakaway: $s = 0$



Exercise 63

Sketch the root loci for the system shown.



Follow the steps given in this lecture.

Exercise 63 - continued

$$H(s) = \frac{s+0.4}{s^2(s+3.6)}$$

$n - m = 2$

$q = 1$ $\quad$ $q = 2$

$\theta_1 = 90^\circ$ $\quad$ $\theta_2 = -90^\circ$

$$\alpha = \frac{\sum p - \sum z}{n-m}$$

$$\alpha = \frac{0 + 0 - 3.6 - (-0.4)}{2} = -1.6$$

Break away point

$1 + K \frac{s+0.4}{s^2(s+3.6)} = 0$ $\quad$ $K = p(s)$

$$p(s) = -\frac{s^2(s+3.6)}{s+0.4}$$

$$\frac{dp(s)}{ds} = \frac{(3s^2 + 7.2s)(s+0.4) - (s^3 + 3.6s^2)}{(s+0.4)^2} = 0$$

$$s^3 + 2.4s^2 + 1.44s = 0$$

$$s(s+1.2)^2 = 0$$

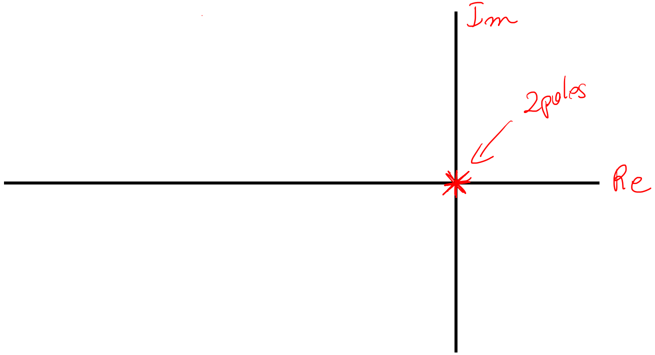
$s = 0$

$s = -1.2$

Exercise 63 - continued

$$H(s) = \frac{s + 0.4}{s^2(s + 3.6)}$$

Try to draw the root-locus here before looking at the solution in the next slide.

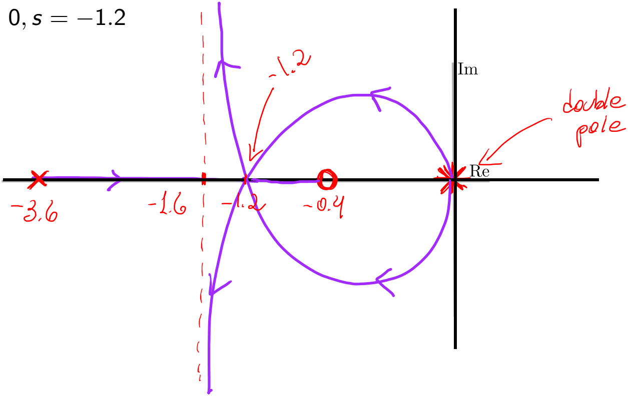


Exercise 63 - continued

$$H(s) = \frac{s + 0.4}{s^2(s + 3.6)}$$

Asymptotes: 90°, -90°, at $s = -1.6$.

Breakaway: $s = 0, s = -1.2$



Exercise 63 - note

An alternative way to find the root locus is to use the angle requirement.

$$\angle\left[k\frac{s+0.4}{s^2(s+3.6)}\right] = 180^\circ + \ell 360^\circ$$

which can be rewritten as

$$\angle(s+0.4) - 2\angle(s) - \angle(s+3.6) = 180^\circ + \ell 360^\circ.$$

Since $s = \sigma + j\omega$

$$\angle(\sigma + j\omega + 0.4) - 2\angle(\sigma + j\omega) - \angle(\sigma + j\omega + 3.6) = 180^\circ + \ell 360^\circ$$

and $\angle s = \tan^{-1}(\omega/\sigma)$, the root locus function satisfies

$$\tan^{-1}\left(\frac{\sigma}{\omega+0.4}\right) - 2\tan^{-1}\left(\frac{\sigma}{\omega}\right) - \tan^{-1}\left(\frac{\sigma}{\omega+3.6}\right) = 180^\circ + \ell 360^\circ$$

The angle of the root locus then is $d\omega/d\sigma$.

Exercise 64

Consider a unit feedback system with an open loop transfer function

$$L(s) = \frac{k}{s^3 + 50s^2 + 500s + 1000}$$

(a) Find the breakaway point on the real axis

(b) Find the asymptote centroid

(c) Find the value of k at the breakaway point

(d) Draw the root locus for $k \geq 0$

Exercise 64 - continued

$$1 + \frac{k}{s^3 + 50s^2 + 500s + 1000} = 0$$

poles
$s = -10$
$s = -37$
$s = -2$

a) $p(s) = -(s^3 + 50s + 500s + 1000)$

$\frac{dp(s)}{ds} = -(3s^2 + 100s + 500) = 0$

$s = -27.2$ (Not on the root locus)

$s = -6.12$

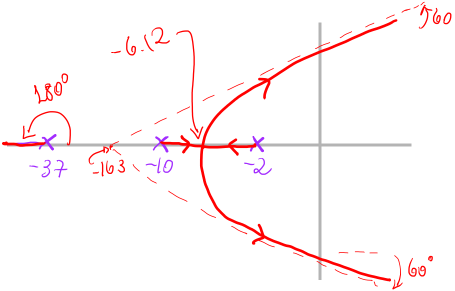
b) $n - m = 3$

$\theta = 60°, 180°, -60°$

$\alpha = \frac{-10 - 37 - 2}{3} = -16.33$

c) $p(-6.12) = 416$

$k = 416$



Exercise 65

Homework

The primary mirror of a large telescope can have a diameter of 10 m and a mosaic of 36 hexagonal segments with the orientation of each segment actively controlled. Suppose this unit feedback system for the mirror segments has the open loop transfer function

$$L(s) = k\frac{1}{s(s^2+2s+5)}.$$

(a) Find the asymptotes and sketch them in the s-plane

(b) Find the angle of departure from the complex poles

(c) Determine the gain when 2 roots lie on the imaginary axis

(d) Sketch the root locus

Exercise 65 - continued

$$1 + k\frac{1}{s(s^2+2s+5)} = 0$$

poles: $s = 0$
$s = -1 + 2j$
$s = -1 - 2j$

(a) Find the asymptotes and sketch them in the s-plane

$n - m = 3$, $q = 1, 2, 3$

$q = 1$: $\theta_1 = 60^0$

$q = 2$: $\theta_2 = 180^0$

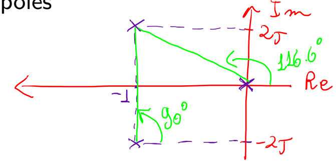
$q = 3$: $\theta_3 = -60^0$

$\alpha = \frac{\sum p - \sum z}{n-m}$, $\alpha = \frac{-1-1-0}{3}$, $\alpha = -\frac{2}{3}$

(b) Find the angle of departure from the complex poles

$90^0 + \phi + 116.6^0 = 180^0$

$\boxed{\phi = -26.6^0}$



Exercise 65 - continued

$$1 + k\frac{1}{s(s^2+2s+5)} = 0$$

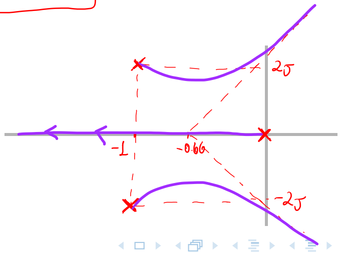
(c) Determine the gain when 2 roots lie on the imaginary axis

s^3	1	5
s^2	2	K
s^1	$5-\frac{K}{2}$	0
s^0	K	0

if $5-\frac{K}{2}=0$, roots lie on the imaginary axis

$K=10$

(d) Sketch the root locus



Next class...

- PID controllers