MECE 3350U
Control Systems

Lecture 10

Routh-Hurwitz Stability Criterion

Outline of Lecture 10

By the end of today's lecture you should be able to

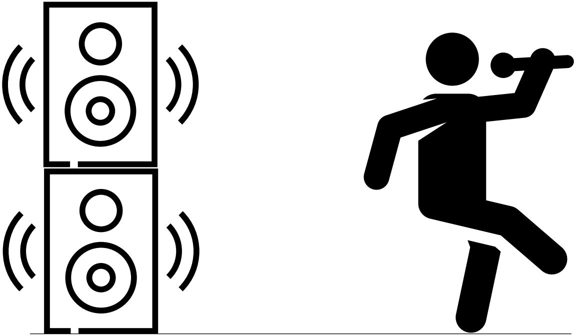
- Understand the principle of stability
- Define the conditions required for stability
- Apply to the Routh-Hurwitz stability criterion

Applications

Are these control systems stable without feedback control?

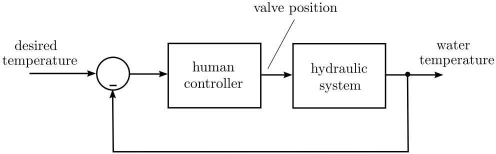
Applications

Example of destabilizing positive feedback.



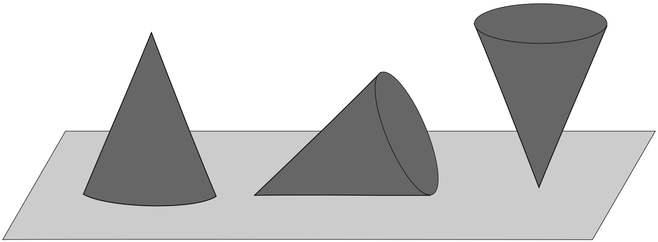
Applications

High control loop gains can make a system unstable.



The concept of stability

A stable system is a dynamic system with a bounded response to a bounded input.



A system is stable is all closed-loop transfer function poles lie in the left-half $s-$plane.

Requirements for stability

Consider the generic transfer function:

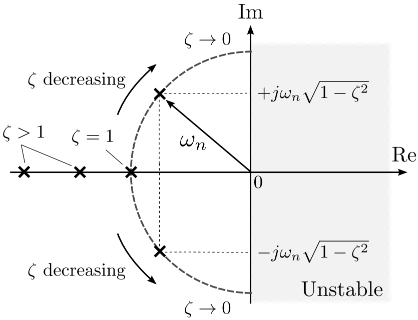
$$T(s) = \frac{p(s)}{q(s)} = \frac{k \prod_{i=1}^{M} (s + z_i)}{s^N \prod_{k=1}^{Q} (s + \sigma_k) \prod_{m=1}^{R} \left[s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)\right]} \qquad (1)$$

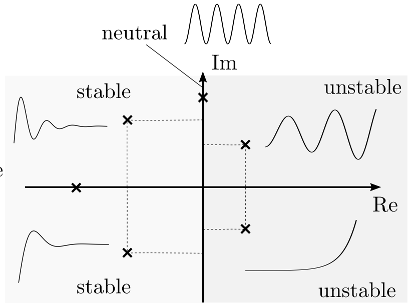
The output response for an impulse function input and $N = 0$ is

$$y(t) = \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} B_m \left(\frac{1}{\omega_m}\right) e^{-\alpha_m t} \sin(\omega_m t + \theta_m) \qquad (2)$$

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have **negative real parts**

Stability and the location of poles





Stable system: Poles are in the left-half plane

Neutral system or marginally stable: Poles are purely imaginary ($j\omega$)

Unstable system: At least one of the poles is in the right-half plane

Marginally stable

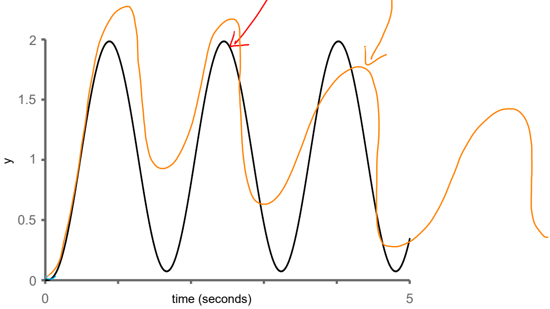
Some poles of the closed-loop transfer functions are purely imaginary.

Example: The transfer function

$$T(s) = \frac{1}{(s+10)(s^2+16)} \tag{3}$$

has the poles $s_1 = -10$, $s_2 = 4j$, $s_3 = -4j$.

For $r(t) = 1$:



The Routh-Hurwitz criterion

This criterion is a necessary and sufficient condition for stability

Order the coefficient of the characteristic equation

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \ldots + a_1 s + a_0 = 0 \tag{4}$$

Into an array as follows:

$$\begin{array}{l|llll} s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{5}$$

The number of roots with positive real pats is equation to the number of changes in sign of the first column.

The Routh-Hurwitz criterion

Step 1: Place the highest order of $q(s)$ on the top-left column from n to 0.

$$\begin{array}{|c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{6}$$

Step 2: From the second column, the first two rows are the coefficients of

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \ldots + a_1 s + a_0 = 0$$

$$\begin{array}{c|cccc|} \cline{2-5} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ \cline{2-5} s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{7}$$

The Routh-Hurwitz criterion

Step 3: Fill out the reminder rows

$$
\begin{array}{lcccc}
s^n & a_n & a_{n-2} & a_{n-4} & \dots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\
s^{n-2} & \boxed{b_{n-1}} & \boxed{b_{n-3}} & \boxed{b_{n-5}} & \boxed{\dots} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\
\vdots & \vdots & \vdots & \vdots & \\
s_0 & h_{n-1} & & &
\end{array}
\tag{8}
$$

For the b_n coefficients:

there is a typo in the original note. Please check.

$$
b_{n-1} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{array} \right\| \tag{9}
$$

$$
b_{n-3} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{array} \right\| \tag{10}
$$

The Routh-Hurwitz criterion

Step 3: Fill out the reminder rows

$$\begin{array}{lllll} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & \boxed{c_{n-1} \quad c_{n-3} \quad c_{n-5} \quad \dots} & & & \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{11}$$

For the c_n coefficients:

Please check for typos in the original note

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{array} \right\| \tag{12}$$

$$c_{n-3} = \frac{-1}{b_{n-1}} \left\| \begin{array}{cc} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{array} \right\| \tag{13}$$

And so on...

Example of Routh-Hurwitz matrix

$$q(s) = s^5 + 2s^4 + 1s^3 + 4s^2 + 11s + 10$$

The Routh-Hurwitz matrix is

$$\begin{array}{c|ccc}
s^5 & 1 & 1 & 11 \\
s^4 & 2 & 4 & 10 \\
\hline
s^3 & -1 & 6 & 0 \\
\hline
s^2 & 16 & 10 & 0 \\
\hline
s^1 & 6.25 & 0 & 0 \\
\hline
s^0 & 10 & 0 & 0
\end{array}$$

$$\frac{1\times4-1\times2}{-2}$$

$$\frac{1\times10-11\times2}{-2}$$

$$\frac{2\times6-4(-1)}{-(-1)}$$

$$\frac{2\times0-(-1)(10)}{-(-1)}$$

$$\frac{-1(10)-6(16)}{16}$$

$\to$ Unstable

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{Vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{Vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{Vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{Vmatrix}$$

$$\frac{16\times0-10\times6.25}{-6.25}$$

Example of Routh-Hurwitz matrix

$$q(s) = 3s^5 + s^4 + 2s^3 + 1s^2 + 0s + 1$$

The Routh-Hurwitz matrix is

$\dfrac{3\times1-2\times1}{-1}$

$\dfrac{3\times1-0\times1}{-1}$

$\dfrac{1(-3)-1(-1)}{-(-1)}$

$\dfrac{1\times0-1(-1)}{-(-1)}$

$\dfrac{(-1)(1)-(-3)(-2)}{-(-2)}$

$\dfrac{-2(0)-1(-3.5)}{-(-3.5)}$

s^5	3	2	0
s^4	1	1	1
s^3	−1	−3	0
s^2	−2	1	0
s^1	−3.5	0	0
s^0	1		

unstable

$$b_{n-1} = \frac{-1}{a_{n-1}} \left\| \begin{matrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{matrix} \right\|$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{matrix} a_{n-1} & a_{n-3} \\ b_{n-1} & a_{n-3} \end{matrix} \right\|$$

The Routh-Hurwitz Criterion

Stability requires all roots of $q(s)$ to have positive real parts.

$\rightarrow$ Count the sign *changes* in the first column

$\rightarrow$ That is the number of roots in the right half plane

Case 1: All elements in the first column are nonzero

$$q(s) = a_2 s + a_1 s + a_0$$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s_1 & a_1 & 0 \\ s_0 & a_0 & 0 \end{array}$$

Thus the system is stable if:

$\rightarrow$ $a_2 > 0$, **and** $a_1 > 0$, **and** $a_0 > 0$.

More likely

or

$\rightarrow$ $a_2 < 0$, **and** $a_1 < 0$, **and** $a_0 < 0$.

The Routh-Hurwitz Criterion

Case 2: There is a zero in the first column. Other elements in the row containing the zero are nonzero.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^{3} + 11s + 10$$

Replace the zero in the first column with $\epsilon \to 0^+$, i.e. $0 < \epsilon << 1$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s_1 & d_1 & 0 & 0 \\ s_0 & 10 & 0 & 0 \end{array}$$

sign change

$$\lim_{\epsilon \to 0} c_1 = -\infty$$

$$\lim_{\epsilon \to 0} d_1 = +\infty$$

$\boxed{\text{unstable}}$

where:

$$c_1 = \frac{4\epsilon - 12}{\epsilon}, \qquad d_1 = \frac{6c_1 - 10\epsilon}{c_1}$$

Now, make $\epsilon \to 0^+$ and evaluate the first column elements signs

The Routh-Hurwitz Criterion

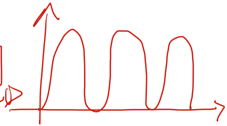
Case 3: There is a zero in the first column and the other elements of the row containing the zero are also zero.

Case 4: Repeated roots of the characteristic equation on the imaginary axis.

What does it mean?

$\Rightarrow$ $\boxed{\text{The characteristic equations has purely imaginary roots}}$

No exponential



$$
\begin{array}{c|cc}
s^3 & 1 & 4 \\
s^2 & 2 & k \\
s_1 & \frac{8-k}{2} & 0 \\
s_0 & k & 0
\end{array}
$$

if $K=8 \Rightarrow$ Marginally stable

if $K=0 \Rightarrow$ Marginally stable

$\rightarrow$ oscillates indefinitely.

Exercise 45

Find the Routh-Hurwitz matrix for the following closed-loop transfer functions and assess the stability of each system.

Use the provided Matlab code posted on Blackboard to check your answers.

$$T(s) = \frac{1}{s^2 + 2s + 1}$$

$$R(s) = \frac{1 + s}{3s^3 + s^2 + 2s + 3}$$

$$P(s) = \frac{1}{s^4 + 2s^3 - 100s - 500}$$

$$H(s) = \frac{s^2 + 1}{s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63}$$

Exercise 45 - continued

Solution: Use the Matlab script posted on B.B.

$$T(s) = \frac{1}{s^2 + 2s + 1}$$

$$\begin{array}{c|cc} s^2 & 1 & 1 \\ s^1 & 2 & 0 \\ s^0 & 1 & \end{array}$$

the system is stable.

Exercise 45 - continued

Solution: Use the Matlab script posted on B.B.

$$R(s) = \frac{1+s}{3s^3 + s^2 + 2s + 3}$$

Exercise 45 - continued

Solution: Use the Matlab script posted on B.B.

$$P(s) = \frac{1}{s^4 + 2s^3 - 100s - 500}$$

$+0s^2$

$$\begin{array}{c|ccc}
s^4 & 1 & 0 & -500 \\
s^3 & 2 & -100 & 0 \\
\hline
s^2 & 50 & -500 & \\
s^1 & -80 & & \\
s^0 & 500 & &
\end{array}$$

the system is unstable :(

Exercise 45 - continued

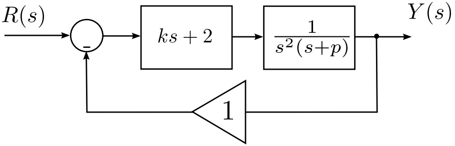
$$H(s) = \frac{s^2 + 1}{s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63}$$

Try this on your own

Solution: Use the Matlab script posted on B.B.

Exercise 46

A closed-loop feedback system is shown in the figure.



For what range of values of the parameters k and p is the system stable ?

Procedure:

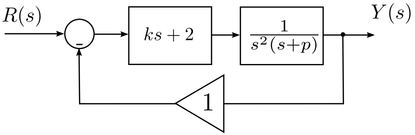
$\rightarrow$ Find the closed-loop transfer function

$\rightarrow$ Write the Routh-Hurwitz matrix

$\rightarrow$ Determine the values of p and k that meet the stability condition

Exercise 46 - continued

Step 1 - Find the closed-loop transfer function



$$\frac{Y(s)}{R(s)} = \frac{Ks+2}{s^2(s+p) + Ks+2}$$

$$q(s) = s^3 + ps^2 + Ks + 2$$

characteristic equation

Exercise 46 - continued

Step 2 - Find the Routh-Hurwitz matrix

$$T(s) = \frac{ks+2}{s^2(s+p)+ks+2} = \frac{ks+2}{s^3+ps^2+ks+2}$$

$$\begin{array}{c|cc}
s^3 & 1 & k \\
s^2 & p & 2 \\
s^1 & \frac{kp-2}{p} & 0 \\
s^0 & 2 &
\end{array}$$

Note: if $kp=2$ the system is marginally stable. Why?

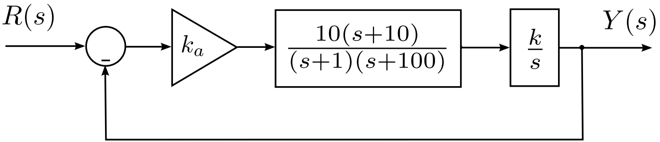
for stability we need

$$\frac{kp-2}{p} > 0 \quad , \quad p > 0$$

$$\boxed{\text{thus} \begin{cases} p > 0 \\ kp > 2 \end{cases}}$$

Exercise 47

The linear model of a phase detector can be represented by the diagram shown. It is designed to maintain a zero phase between the input carrier signal and local voltage controller oscillator. We want to minimize th steady-state error for a ramp input.

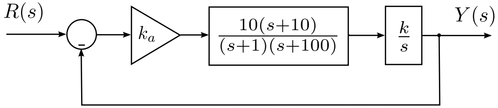


(a) Determine the maximum gain $k_a k$ in order to maintain a stable system.

(b) Find kk_a for a steady-state error of 1° for a ramp signal of 100 rad/s.

Exercise 47 - continued

Step 1 - Find the closed-loop characteristic equation.



$$T(s) = \frac{Y(s)}{R(s)} = \frac{10 K_a K (s+10)}{s(s+1)(s+100) + 10 K K_a (s+10)}$$

$$\boxed{q(s) = s^3 + 101 s^2 + (100 + 10 K K_a) s + 100 K K_a}$$

Exercise 47 - continued

Step 2: Find the Routh matrix

$$q(s) = s^3 + 101s^2 + (100 + 10kk_a)s + 100kk_a = 0$$

$$b = \frac{100\,KKa - 10(100 + 10KKa)}{-101}$$

$$\begin{array}{c|cc} s^3 & 1 & 100 + 10KKa \\ s^2 & 101 & 100\,KKa \\ s^1 & b & 0 \\ s^0 & 100\,KKa & \end{array}$$

for stability:

$b > 0 \rightarrow 100 + \frac{910}{101} KKa > 0$

$$\boxed{KKa > \frac{-100(101)}{910}}$$

$\boxed{\text{and}}$

$100\,KKa > 0$ thus $\boxed{KKa > 0}$ sufficient condition

Exercise 47 - continued

(c) Find kk_a that yield a tracking error of 1° for $r(t) = 100t$

$$e(s) = \lim_{s\to 0} s(1 - T(s))\frac{100}{s^2}$$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{10\,K_a K(s+10)}{s(s+1)(s+100) + 10KK_a(s+10)}$$

$$e_{ss} = \lim_{s\to 0} \cancel{s}\left(\frac{\cancel{s}(s+1)(s+100) + \cancel{10KK_a(s+10)} - \cancel{10KK_a(s+10)}}{s(s+1)(s+100) + 10KK_a(s+10)}\right)\frac{100}{\cancel{s^2}}$$

$$e_{ss} = \frac{100}{KK_a}$$

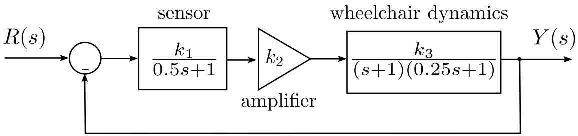
$$e_{ss} < 1$$

$$\frac{100}{KK_a} < 1$$

$$\boxed{KK_a > 100}$$

Exercise 48

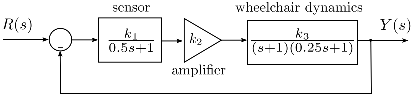
A wheelchair velocity control system is shown in the diagram.



Determine the maximum gain $k_1 k_2 k_3$ for a stable system.

Exercise 48 - continued

The closed loop transfer function



$$\frac{Y(s)}{R(s)} = \frac{K_1K_2K_3}{(s+1)(0.25s+1)(0.5s+1)+K_1K_2K_3}$$

$$\boxed{K = K_1K_2K_3}$$

$$q(s) = 0.125s^3 + 0.875s^2 + 1.75s + 1 + K$$

Exercise 48 - continued

$$q(s) = 0.125s^3 + 0.875s^2 + 1.75s + 1 + K$$

The Routh matrix is

$$\begin{array}{c|cc} s^3 & 0.125 & 1.75 \\ s^2 & 0.875 & 1+K \\ s^1 & b & 0 \\ s^0 & 1+K & 0 \end{array}$$

$$b = \frac{0.125(1+K) - 1.75 \times 0.875}{-0.875}$$

$$b = 1.60714 - 0.142857K$$

Stability requires that:

$1+K > 0$

$\boxed{K > -1}$

$b > 0$

$1.607 - 0.143K > 0$

$\boxed{K < 11.23}$

thus

$$\boxed{-1 < K < 11.25}$$

Exercise 49

A teleoperated control system incorporates both a person and a remote machine. In the case of remote operation of a robot, force feedback is useful. The characteristic equation of such a system is

Homework
try this on your own.

$$s^4 + 20s^3 + k_1 s^2 + 4s + k_2 = 0$$

where k_i is a feedback force amplification factor. Determine and plot the region of stability for this system for k_1 and k_2.

Exercise 49 - continued

$$s^4 + 20s^3 + k_1 s^2 + 4s + k_2 = 0$$

The Routh array is

$$\begin{array}{c|l} s^4 & \\ s^3 & \\ s^2 & \\ s^1 & \\ s^0 & \end{array}$$

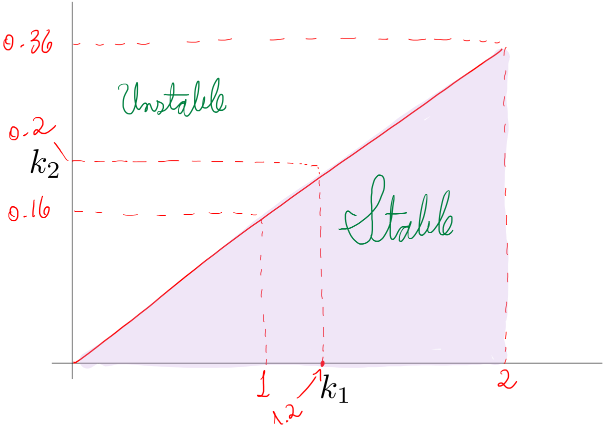
Answer: $K_2 > 0$ and $\dfrac{20K_1 - 4 - 100K_2}{5K_1 - 1} > 0$

$K_1 > 0.2$

combined: $K_2 < 0.2K_1 - 0.04$

Exercise 49 - continued

For stability: $k_2 > 0$, $k_1 > 0.2$, and $k_2 < 0.2k_1 - 0.04$



Next class...

- Root-locus method